MA5210 Differentiable Manifolds – Essay

Peaks, Pits, Passes: A First Look at Morse Theory

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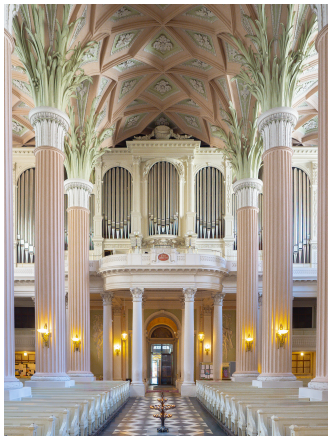
Marston Morse in *Peaks, Pits, Passes* (1965) [13].

Morse theory was first introduced by Harold Calvin Marston Morse (1892—1977) in his 1925 paper *Relations between the critical points of a real function of n independent variables* [11, 15]. Throughout the rest of his career, he would go on to develop what is today known as Morse theory under the guise of 'calculus of variations in the large', publishing the seminal monograph by the same name [12] in 1934 based on lectures delivered to the American Mathematical Society three years earlier.

Since then, Morse's theory of critical points has developed and flourished in a myriad different directions. René Thom made significant contributions in the 40's, followed by Raoul Bott's generalised Morse–Bott functions, which were used to provide the first proof of his 1957 theorem on the periodicity of the homotopy groups of the classical groups $U(\infty)$ and $O(\infty)$ [4, 15]. (This would go on to be both the inspiration and the theoretical backbone for topological K-theory and index theory as conceived by Sir Michael Atiyah, Friedrich Hirzebruch, and others.)

In the 60's, Stephen Smale would also adapt ideas from Morse theory to dynamical systems, where he would use so-called Morse–Smale systems to construct self-indexing Morse functions, whose values at critical points equal the Morse index of those critical points [18]. These functions were key ingredients to Smale's 1967 proof of the generalised Poincaré conjecture in dimension above 4 [19], and of the consequential h-cobordism theorem the next year [20].

The list of landmark achievements indebted to Morse theory does not end there: there is also the Thom–Smale–Witten complex, which leads naturally to Floer homology in symplectic geometry (see [3] for an exposition); surgery theory, closely related to the handle attachment operation, nowadays standard in geometric topology; Almgren–Pitts min-max theory, which is used to construct embedded minimal hypersurfaces [16]; Karen Uhlenbeck's extension of Morse theory by perturbation methods to infinite dimensions (i.e., without the Palais–Smale condition) [21]; and plenty more.



St Nicholas Church, Leipzig.

Henri Poincaré was a pervading influence in Morse's mathematics, running through it like a motif of Bach (Morse was a talented classical keyboardist as well, and loved above all the music of J. S. Bach). He would come to describe two approaches to mathematics: one chiefly concerned with building increasingly contrived foundations, the other akin to building a beautiful cathedral. Poincaré was his model of the latter, a true architect who masterfully married geometry, topology, dynamics, and analysis [2]. Looking back at a century of Morse theory and its ramifications, there remains little doubt that Morse managed to accomplish the same harmonious union in his mathematical oeuvre. In this essay, we will be treated to our first glimpses of this *Nikolaikirche* of 20th century mathematics.

1 Introduction

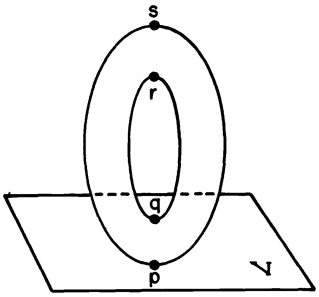


Figure 1. Adapted from [10].

In what follows, we will assume the content of Chapters 1—12, and 14—17 of John M. Lee's *Introduction to Smooth Manifolds* [8], or the lectures of MA5210. Our exposition will largely follow that of the classic [10], beginning with a first example from Part I.

Consider a torus $M := T^2$ embedded in $\mathbb{R}^3$ as pictured in Figure 1 above, and let V be a plane tangent to M at the point p. Let $f : M \to \mathbb{R}$ be the height above V, and let M^a be the set of all points $x \in M$ such that $f(x) \leq a$.

As a varies, we can think of the torus being submerged in a rising sea, and track the space M^a beneath the surface.

i) If $a < f(p)$, then $M^a = \varnothing$.

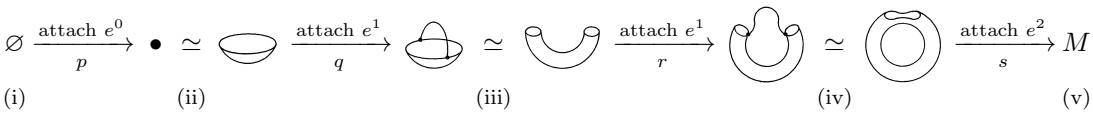
ii) If $f(p) < a < f(q)$, then M^a is homeomorphic to a disc.

iii) If $f(q) < a < f(r)$, then M^a is homeomorphic to a cylinder.

iv) If $f(r) < a < f(s)$, then M^a is homeomorphic to a compact manifold of genus 1 with boundary a circle.

v) If $a > f(s)$, then $M^a = M$.

We can see that something interesting is happening as a passes through the points $f(p)$, $f(q)$, $f(r)$, and $f(s)$. Let us consider the homotopy type of M^a, instead of its homeomorphism type.



What do we mean by 'attaching a k-cell'? Define the k-cell as $e^k := \{x \in \mathbb{R}^k : |x| \leq 1\}$ for $k \geq 0$, with e^0 understood to be a point. Let X be a topological space, and g be a continuous map from $\partial e^k = S^{k-1}$ to X. Then

$$X \cup_g e^k$$

(X *with a* k*-cell attached via* g) is obtained by first taking the disjoint union of X and e^k, then identifying each point of e^k with its image under g in X.

So going from (i) to (ii) consists of attaching a 0-cell; (ii) $\to$ (iii) is attaching a 1-cell; and so on, changing the homotopy type of M^a at each of p, q, r, and s.

It turns out that these are precisely the critical points of the function f. Choosing local coordinates (x, y) around these points, we see that $\partial f/\partial x$ and $\partial f/\partial y$ both vanish. At p, we can choose (x, y) such that $f = x^2 + y^2$. At q or r we can choose (x, y) such that $f = c + x^2 - y^2$, where $c = f(q)$ or $f(r)$, respectively. Finally at s, we can choose (x, y) such that $f = -x^2 - y^2$.

Surprisingly, the number of minus signs in each of the coordinate representations of f is equal to the dimension of the k-cell we attach!

We shall now generalise these facts for any smooth function on a smooth manifold.

2 Morse Functions

2.1 Definitions and Lemmas

Suppose $F : M \to N$ is a smooth map between smooth manifolds, and let $dF_p : T_pM \to T_{F(p)}N$ be its differential at $p \in M$. Recall the following definitions:

Definition 2.1.

i) A point $p \in M$ is said to be a *critical point* of F if

$$\operatorname{rank} dF_p < \min\{\dim M, \dim N\}.$$

It is called a *regular point* of F if it is not a critical point, i.e., dF_p has full rank. The set of all critical points of F is called the *critical set* of F, which we denote by $\mathbf{Cr}_F$.

ii) $q \in N$ is called a *critical value* of F if the level set $F^{-1}(q)$ contains a critical point of F, or else a *regular value* if it is not a critical value.

Theorem 2.2 (Sard's Theorem)**.** *Suppose $F : M \to N$ is a smooth map. Then the set of critical values of F has Hausdorff dimension*[1] *at most* $\dim N - 1$*. In particular, it has measure zero in N, and the set of regular values of F is dense in N.*

For a proof, see [8, Ch. 6].

Since we will mostly be interested in smooth real-valued functions f on M, note that in this case, p is a critical point of f precisely when df_p is zero, i.e., given any smooth coordinate chart $(U, (x^1, \cdots, x^n))$ containing p, $\frac{\partial f}{\partial x^1}(p) = \cdots = \frac{\partial f}{\partial x^n}(p) = 0$.

Given tangent vectors $v, w \in T_pM$, we can extend these to vector fields $\tilde{v}, \tilde{w}$ on M.

Definition 2.3. Let $f : M \to \mathbb{R}$ be a smooth function, and let p be a critical point of f. The *Hessian of f at p* is the symmetric bilinear form $H_{f,p} : T_pM \times T_pM \to \mathbb{R}$ defined by

$$H_{f,p}(v, w) := (\tilde{v}\tilde{w}f)(p) = \tilde{v}_p(\tilde{w}(f)).$$

We must check that the Hessian is in fact symmetric and well-defined. It is symmetric because $(\tilde{v}\tilde{w}f)(p) - (\tilde{w}\tilde{v}f)(p) = [\tilde{v}, \tilde{w}]_p f = 0$, where $[\cdot, \cdot]$ denotes the Lie bracket.

Since the last expression in the definition of the Hessian involves $\tilde{v}_p = v$, it does not matter what vector field $\tilde{v}$ we choose. Similarly, since $H_{f,p}$ is symmetric, $H_{f,p}(v, w) = \tilde{w}_p(\tilde{v}(f))$, and our choice of $\tilde{w}$ does not matter either. Hence, $H_{f,p}$ is well-defined.

More concretely, if (x^i) are local coordinates, then writing $v := \sum_i v^i \frac{\partial}{\partial x^i}|_p$, $w := \sum_j w^j \frac{\partial}{\partial x^j}|_p$, we can take the extension $\tilde{w} := \sum_j w^j \frac{\partial}{\partial x^j}$, where w^j is a constant. Then

$$H_{f,p}(v, w) = v(\tilde{w}(f)) = v\left(\sum_j w^j \frac{\partial f}{\partial x^j}\right) = \sum_{i,j} v^i w^j \frac{\partial^2 f}{\partial x^i \partial x^j}(p),$$

so that the *Hessian matrix* $(\frac{\partial^2 f}{\partial x^i \partial x^j}(p))$ represents the bilinear form $H_{f,p}$ with respect to the basis $\{\frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x^n}|_p\}$.

Definition 2.4.

i) The *index* of a bilinear form $H : V \times V \to \mathbb{R}$ is the maximum dimension of a subspace W of V such that H is negative-definite on W ($H(w, w) < 0$ for all $w \in W$).

ii) The *nullity* of a bilinear form $H : V \times V \to \mathbb{R}$ is the dimension of the *nullspace* of H, i.e., the subspace of all $u \in V$ such that $H(u, v) = 0$ for all $v \in V$.

iii) A critical point p of $f : M \to \mathbb{R}$ is said to be *nondegenerate* if the nullity of $H_{f,p}$ is zero. Equivalently, p is a nondegenerate critical point iff the Hessian matrix of f at p is invertible.

[1] See also [17] for a more complete version of the Hausdorff dimension statement.

We will refer to the index of $H_{f,p} : T_p \times T_p \to \mathbb{R}$ as just the *index of f at p*, and denote it by $\lambda(f,p)$. The following result will show that the behaviour of f near a nondegenerate critical point p can be characterised by its index.

Lemma 2.5 (Morse Lemma)**.** *Let p be a nondegenerate critical point of the smooth function $f : M \to \mathbb{R}$. Then there exists a chart (U, φ) containing p with local coordinates $(x^1, \cdots, x^n)$ such that*

$$f|_U = f(p) - \sum_{i=1}^{\lambda} (x^i)^2 + \sum_{j=\lambda+1}^{n} (x^j)^2,$$

where $\lambda := \lambda(f,p)$.

Before we prove the Morse lemma, we will need a helper lemma.

Lemma 2.6. *Suppose f is a smooth function defined in a star-shaped neighbourhood $U \subseteq \mathbb{R}^n$ of 0, with $f(0) = 0$. Then there exist smooth functions g_i defined on U such that*

$$f(x^1, \cdots, x^n) = \sum_{i=1}^{n} x^i g_i(x^1, \cdots, x^n),$$

with $g_i(0) = \frac{\partial f}{\partial x^i}(0)$.

Proof.

$$f(x^1, \cdots, x^n) = \int_0^1 \frac{df(tx^1, \cdots, tx^n)}{dt}\, dt = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(tx^1, \cdots, tx^n) \cdot x^i \, dt.$$

Then define $g_i := \int_0^1 \frac{\partial f}{\partial x^i}(tx^1, \cdots, tx^n)\, dt$. □

Proof of the Morse lemma. First, we show that if any such expression for f on U exists, then λ must be equal to the index of f at p. We have that

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(p) = \begin{cases} -2 & \text{if } i = j \le \lambda, \\ 2 & \text{if } i = j > \lambda, \\ 0 & \text{if } i \ne j. \end{cases}$$

Hence the Hessian matrix of f at p is a diagonal matrix with the first λ diagonal entries equal to -2, and the remaining $n - \lambda$ entries equal to 2. Thus $H_{f,p}$ is negative-definite on $\operatorname{span}\{\frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x^\lambda}|_p\}$. If there exists a subspace W of dimension greater than λ, then it must contain some linear combination of the $\frac{\partial}{\partial x^{\lambda+1}}|_p, \cdots, \frac{\partial}{\partial x^n}|_p$, on which $H_{f,p}$ is positive-definite. Thus, $\lambda = \lambda(f,p)$.

Next, we can assume that $\varphi(p) = 0$, so that $\hat{f} := f \circ \varphi^{-1}$ is defined in a neighbourhood $\varphi(U)$ of the origin[2]; and furthermore, assume $\hat{f}(0) = f(p) = 0$. Then by our helper lemma, there exist smooth functions g_i on $\varphi(U)$ such that $\hat{f} = \sum_{i=1}^n x^i g_i$, and $g_i(0) = \frac{\partial \hat{f}}{\partial x^i}(0) = \frac{\partial f}{\partial x^i}(p) = 0$, since p is a critical point of f. Applying the lemma again to g_i, we get $g_i = \sum_{j=1}^n x^j h_{ij}$ for some smooth h_{ij}. Thus,

$$\hat{f}(x^1, \cdots, x^n) = \sum_{1 \le i,j \le n} x^i x^j h_{ij}(x^1, \cdots, x^n).$$

We may assume that $h_{ij} = h_{ji}$, since we can write $\tilde{h}_{ij} := \frac{1}{2}(h_{ij} + h_{ji})$. Furthermore,

$$\begin{aligned} h_{ij}(0) &= \frac{\partial g_i}{\partial x^j}(0) \\ &= \frac{\partial}{\partial x^j}\bigg|_0 \left(\int_0^1 \frac{\partial f}{\partial x^i}(tx^1, \cdots, tx^n)\, dt \right) \\ &= \int_0^1 \frac{\partial f^2}{\partial x^i \partial x^j}(0) t \, dt = \frac{1}{2} \frac{\partial f^2}{\partial x^i \partial x^j}(0), \end{aligned}$$

[2]Note that $\varphi(U)$ should be star-shaped, but this is easily handled.

so the matrix $(\tilde{h}_{ij}(0))$ is equal to $(\frac{1}{2}\frac{\partial^2 \hat{f}}{\partial x^i \partial x^j}(0))$ and is invertible.

Now proceed via induction. Suppose there exists a chart $(U_1, (u^i))$ containing p such that

$$\hat{f} = \pm(u^1)^2 \pm \cdots \pm (u^{k-1})^2 + \sum_{i,j \geq k} u^i u^j H_{ij}(u^1, \cdots, u^n).$$

on U_1, where $H_{ij} = H_{ji}$. After a linear transformation of the last $n-k+1$ coordinates, we may assume that $H_{kk}(0) \neq 0$. Let $G(u^1, \cdots, u^n) := \sqrt{|H_{kk}(u^1, \cdots, u^n)|}$, which is a smooth nonzero function in a neighbourhood $U_2 \subseteq U_1$ of 0. We construct new coordinates (v^i) by defining

$$v^k(u^1, \cdots, u^n) := G(u^1, \cdots, u^n)\left(u^k + \sum_{i>k} u^i H_{ik}(u^1, \cdots, u^n)/H_{kk}(u^1, \cdots, u^n)\right),$$

and $v^i := u^i$ for all other $i \neq k$. By the inverse function theorem, (v^i) are coordinates in some neighbourhood $U_3 \subseteq U_2$ of p. Finally, we can check that

$$\hat{f} = \pm(v^1)^2 \pm \cdots \pm (v^k)^2 + \sum_{i,j>k} v^i v^j H'_{ij}(v^1, \cdots, v^n).$$

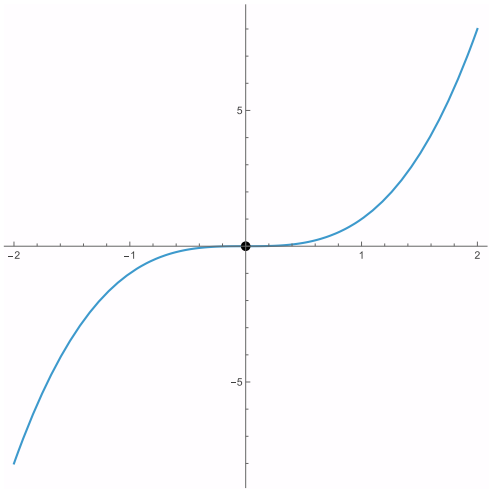
This completes our induction argument. □

Corollary 2.7. *Nondegenerate critical points are isolated.*

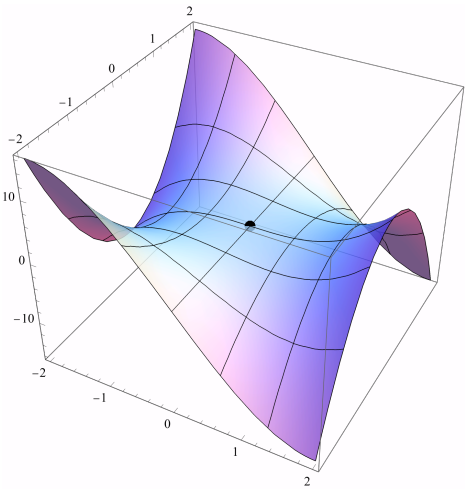
Proof. In the expression obtained for f, we see that in a neighbourhood of p, $\frac{\partial f}{\partial x^i} = \pm 2x^i$ for all i, so $\frac{\partial f}{\partial x^1}(q) = \cdots = \frac{\partial f}{\partial x^n}(q) = 0$ iff $(x^1(q), \cdots, x^n(q)) = 0$ iff $q = p$. □

Example 2.8. We list a few examples of degenerate critical points for functions on $\mathbb{R}$ and $\mathbb{R}^2$ to get a feel for their behaviour.

1) $f(x) = x^3$ — The origin is a degenerate critical point.
2) $f(x) = e^{-1/x^2}\sin^2(1/x)$ — The origin is a nonisolated, hence degenerate, critical point.
3) (*monkey saddle*) $f(x, y) = x^3 - 3xy^2$ — The origin is a degenerate critical point.
4) $f(x, y) = x^2y^2$ — The critical set of f is the union of the x- and y-axes. All critical points are nonisolated, so of course degenerate.



Example 2.8.1. $f(x) = x^3$



Example 2.8.3. $f(x, y) = x^3 - 3xy^2$

Definition 2.9. A smooth function $f : M \to \mathbb{R}$ is called a *Morse function* if all its critical points are nondegenerate.

2.2 Existence of Morse Functions

Are Morse functions abundant 'in nature'? In this section, we describe in brief a proof that they form a dense open subset in the space of smooth functions.

Suppose M is a connected smooth m-manifold. By Whitney's embedding theorem, we can assume that M is embedded in $\mathbb{R}^n, n \leq 2m+1$. Suppose N is a smooth manifold and $F : N \times \mathbb{R}^n \to \mathbb{R}$ is a smooth function. We regard F as a smooth family of functions parametrised by N:

$$F_\nu : \mathbb{R}^n \to \mathbb{R}; \quad F_\nu(x) := F(\nu, x)$$

for each $\nu \in N$. Let $f := F|_{N\times M}, f_\nu := F_\nu|_M$. For each $p \in M$, there is a surjective linear map $\rho_p : (\mathbb{R}^n)^* \to T_p^*M$ which sends each linear functional on $\mathbb{R}^n$ to its restriction to $T_pM \subseteq \mathbb{R}^n$. In particular, we have

$$d(f_\nu)_p = \rho_p(d(F_\nu)_p).$$

We also have a smooth map

$$\partial^p f : N \to T_p^*M; \quad \nu \mapsto d(f_\nu)_p.$$

Definition 2.10.

i) The family $F : N \times \mathbb{R}^n \to \mathbb{R}$ is said to be *sufficiently large relative to the submanifold* $M \hookrightarrow \mathbb{R}^n$ if $\dim N \geq \dim M$ and for every $p \in M$, the point $0 \in T_p^*M$ is a regular value for $\partial^p f$.

ii) F is said to be *large* if for every $x \in \mathbb{R}^n$, the map

$$\partial^x F : N \to (\mathbb{R}^n)^*; \quad \nu \mapsto d(F_\nu)_x$$

is a submersion.

Proposition 2.11. *If $F : N \times \mathbb{R}^n \to \mathbb{R}$ is large, then it is sufficiently large relative to any submanifold $M \hookrightarrow \mathbb{R}^n$.*

Proof. $\partial^p f = \rho_p(d(F_\nu)_p)$ is the composition of two submersions, so is itself a submersion. In particular, it has no critical values. □

Theorem 2.12. *If the family $F : N \times \mathbb{R}^n \to \mathbb{R}$ is sufficiently large relative to the submanifold $M \hookrightarrow \mathbb{R}^n$, then the set of all $\nu \in N$ such that f_ν is not a Morse function has measure zero in N.*

Proof. Our strategy will be to construct a smooth map $\pi : Z \to N$, so that for regular values $\nu \in N$ of this map, f_ν is a Morse function. We can then apply Sard's theorem.

First, let us deal with the special case where there exist global coordinates $(x^1, \cdots, x^n)$ on $\mathbb{R}^n$ (not necessarily linear coordinates) such that M can be identified with an open subset of the coordinate 'plane' $x^{m+1} = \cdots = x^n = 0$. Call such an M a *special* manifold.

Let $\varphi_\nu : M \to \mathbb{R}^m$ be defined as

$$\varphi_\nu(x^1, \cdots, x^m) := \left(\frac{\partial f_\nu}{\partial x^1}(p), \cdots, \frac{\partial f_\nu}{\partial x^m}(p)\right), \text{ where } p := (x^1, \cdots, x^m) \in M.$$

In this setup, the point $p \in M$ is a nondegenerate critical point of f_ν if:

- $\varphi_\nu(p) = 0$, and
- the differential $d(\varphi_\nu)_p : T_pM \to \mathbb{R}^m$ is bijective.

Thus, f_ν is a Morse function iff 0 is a regular value of φ_ν.

Now consider the family $\Phi : N \times M \to \mathbb{R}^m$; $\Phi(\nu, p) := \varphi_\nu(p)$.

Lemma. *0 is a regular value of Φ.*

Proof of lemma. Suppose $(\nu_0, p) \in N \times M$ such that $\Phi(\nu_0, p) = \varphi_{\nu_0}(p) = 0$. Since the family F is sufficiently large, the differential $d(\partial^p f)_{\nu_0}$ is a surjection, so it is not difficult to verify that

$$d_\nu \Phi_{\nu_0} = \left.\frac{\partial}{\partial \nu}\right|_{\nu=\nu_0} d(f_\nu)_p$$

is surjective. Therefore, the differential $d\Phi_{(\nu_0,p)} : T_{(\nu_0,p)}(N \times M) \to \mathbb{R}^m$ itself is surjective. □

It follows that $Z := \Phi^{-1}(0)$ is a closed smooth submanifold of $N \times M$. The natural projection $\pi : N \times M \to N$ restricts to a smooth map $\pi : Z \to N$.

Suppose that ν is a regular value of π. Our goal is to prove that for every $p \in M$ satisfying $\varphi_\nu(p) = 0$, the differential $d(\varphi_\nu)_p : T_pM \to \mathbb{R}^m$ is surjective.

Set[3]

$$T_1 := T_\nu N, \quad T_2 := T_pM,$$
$$D_1 := d_\nu \Phi_\nu : T_1 \to \mathbb{R}^m, \quad D_2 := d(\varphi_\nu)_p : T_2 \to \mathbb{R}^m.$$

Then $d\Phi_{(\nu,p)} = D_1 + D_2$ is surjective by the lemma above, and $T_{(\nu,p)}Z = \ker(D_1 + D_2)$. Let $x \in \mathbb{R}^m$. Then there exists $(t_1, t_2) \in T_1 \oplus T_2$ such that $x = D_1t_1 + D_2t_2$. At the same time, for each $t_1 \in T_1$, there exists $t_2' \in T_2$ such that $(t_1, t_2') \in \ker(D_1 + D_2)$. Then

$$x = (D_1t_1 + D_2t_2) - (D_1t_1 + D_2t_2') = D_2(t_2 - t_2').$$

Since x was arbitrary, this proves that $D_2 = d(\varphi_\nu)_p$ is surjective. This implies that 0 is a regular value of φ_ν, so f_ν is indeed a Morse function. Therefore, by Sard's theorem (Thm. 2.2), the set of $\nu \in N$ such that f_ν is not Morse has measure zero in N.

The general case follows from taking a countable open cover (U_i) of M such that each U_i is special. Let $N_i \subseteq N$ be the set of all $\nu \in N$ such that the restriction of f_ν to U_i is not Morse. Then the countable union $\bigcup_i N_i$ is a set of measure zero in N, and for all ν outside this set, f_ν is Morse. □

Thus, a Morse function is a 'typical' function on a manifold. We can define several kinds of Morse functions, such as *perfect*, *excellent*, or the aforementioned *self-indexing* Morse functions, each of which are extremely useful in various settings.

2.3 Critical Points and Homotopy Type

As before, if f is a smooth real-valued function on a smooth manifold M, we define the *sublevel set*

$$M^a := f^{-1}((-\infty, a]) = \{p \in M : f(p) \leq a\}.$$

By the implicit function theorem, M^a is a smooth manifold with boundary whenever a is not a critical value of f.

We thus begin the study of Morse theory proper by formulating two principles, one 'weak' and one 'strong'. The former will state that the homotopy type of a sublevel set does not change as we pass through regular values of a smooth function. The latter will tell us that passing through a critical point of index λ corresponds precisely to attaching a λ-cell.

Theorem 2.13 (Weak Morse Principle)**.** *Let f be a smooth real-valued function on a manifold smooth M. Suppose $f^{-1}([a, b])$ is compact and contains no critical points of f. Then the sublevel sets M^a and M^b are diffeomorphic. Furthermore, M^a is a deformation retract of M^b, so that the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence.*

Proof. Choose a Riemannian metric $\langle \cdot, \cdot \rangle$ on M. (For basics on Riemannian metrics, see [8, Ch. 13].) The *gradient* of f is the vector field ∇f on M that satisfies $\langle X, \nabla f \rangle = Xf$ for any vector field X. We have that $\nabla f_p = 0$ precisely at the critical points p of f.

[3]Forgive the slight abuse of notation in defining D_1.

Construct a smooth function $\rho : M \to \mathbb{R}$ equal to $-1/\langle \nabla f, \nabla f \rangle$ on the whole compact set $f^{-1}([a,b])$, and supported in a compact neighbourhood of $f^{-1}([a,b])$. Now define the vector field X by $X_q := \rho(q)\nabla f_q$, and let $\Phi : \mathbb{R} \times M \to M; (t,x) \mapsto \Phi_t(x)$ be the flow generated by X. Let $u(t)$ be an integral curve of X; differentiating f along $u(t)$, we have that

$$\frac{df}{dt} = Xf = -\frac{1}{\langle \nabla f, \nabla f \rangle} \langle \nabla f, \nabla f \rangle = -1.$$

This implies that $\Phi_{b-a}(M^b) = M^a$ and $\Phi_{a-b}(M^a) = M^b$, so Φ_{b-a} is the diffeomorphism desired.

Now for $t \in [0,1]$, define

$$\Psi_t : M^b \to M^b; \Psi_t(q) := \begin{cases} q & \text{if } f(q) \leq a \\ \Phi_{t(f(q)-a)}, & \text{if } a \leq f(q) \leq b. \end{cases}$$

Then Ψ_0 is the identity, and Ψ_1 is a retraction from M^b to M^a. This proves that M^a is a deformation retract of M^b. □

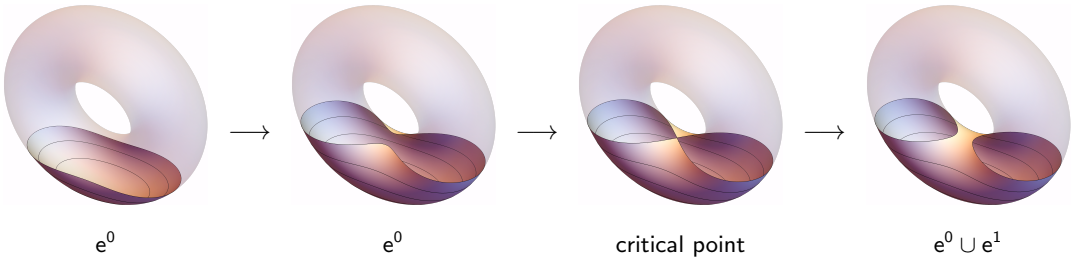


Figure 2

Returning to our initial example of the torus, we see that the first sublevel set in Figure 2 is indeed a deformation retract of the second. This next theorem describes the situation before, at, and after the third sublevel set.

Theorem 2.14 (Strong Morse Principle)**.** *Let f be a smooth real-valued function on M, and let p be a nondegenerate critical point of f of index λ. Set $c := f(p)$. Suppose $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and contains no critical point of f other than p for some $\varepsilon > 0$. Then for $\varepsilon > 0$ small enough, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached.*

Proof. Choose a smooth chart (U, φ) containing p with local coordinates (u^i) so that on U, f is in the form given by the Morse lemma:

$$f = c - \sum_{i=1}^{\lambda} (u^i)^2 + \sum_{j=\lambda+1}^{n} (u^j)^2.$$

The critical point p has coordinates $u^1(p) = \cdots = u^n(p) = 0$. Choose $\varepsilon > 0$ small enough that

- $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and contains no critical point of f besides p (we can do this by Cor. 2.7), and
- the image $\varphi(U) \subseteq \mathbb{R}^n$ contains the closed ball $\overline{B_{\sqrt{2\varepsilon}}(0)} = \{(u^1, \cdots, u^n) : \sum_{i=1}^n (u^i)^2 \leq 2\varepsilon\}$.

Let the λ-cell e^λ be the set of points in U such that

$$(u^1)^2 + \cdots + (u^\lambda)^2 \leq \varepsilon \quad \text{and} \quad u^{\lambda+1} = \cdots = u^n = 0.$$

Note that the only points $q \in e^\lambda$ such that $f(q) \leq c - \varepsilon$ are precisely those on the boundary of e^λ, so $e^\lambda \cap M^{c-\varepsilon} = \partial e^\lambda$ and e^λ is attached to $M^{c-\varepsilon}$ as required.

Now let $\mu : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

$$\bullet\ \mu(0) > \varepsilon, \quad \bullet\ \mu(t) = 0 \text{ for } t \geq 2\varepsilon, \quad \bullet\ -1 < \mu' \leq 0, \text{ where } \mu := \tfrac{d\mu}{dt}.$$

Define the function $F : M \to \mathbb{R}$ by

$$F := \begin{cases} f & \text{if } q \in M - U, \\ f - \mu\left(\sum_{i=1}^{\lambda}(u^i)^2 + \sum_{j=\lambda+1}^{n} 2(u^j)^2\right) & \text{if } q \in U. \end{cases}$$

By our constraints, F is easily seen to be smooth. For convenience, let $\xi := \sum_{i=1}^{\lambda}(u^i)^2$ and $\eta := \sum_{j=\lambda+1}^{n}(u^j)^2$. Then for $q \in U$, we have $F = f - \mu(\xi + 2\eta) = c - \xi + \eta - \mu(\xi + 2\eta)$.

We claim that the sublevel sets $F^{-1}((-\infty, c+\varepsilon])$ and $M^{c+\varepsilon} = f^{-1}((-\infty, c+\varepsilon])$ are identical. To see this, note that F and f agree when $\xi + 2\eta \geq 2\varepsilon$. When $\xi + 2\eta < 2\varepsilon$, we have that

$$\begin{aligned} F \leq f &= c - \xi + \eta \\ &\leq c + \frac{1}{2}\xi + \eta \\ &\leq c + \varepsilon. \end{aligned}$$

Moreover, we claim that the critical sets of F and f agree: that is, $\mathbf{Cr}_F = \mathbf{Cr}_f$. It suffices to look only at points within U. We make the following computations:

$$\frac{\partial F}{\partial \xi} = \frac{\partial}{\partial \xi}(c - \xi + \eta - \mu(\xi + 2\eta)) = -1 - \mu'(\xi + 2\eta) < 0,$$

$$\frac{\partial F}{\partial \eta} = 1 - 2\mu'(\xi + 2\eta) \geq 1.$$

Since $dF = \frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta$, and the covectors $d\xi$ and $d\eta$ both vanish only at the origin, the only critical point both F and f have in U is p.

The set $F^{-1}([c-\varepsilon, c+\varepsilon])$ is contained in $f^{-1}([c-\varepsilon, c+\varepsilon])$ and is therefore compact. It contains no critical points except possibly p. But $F(p) = c - \mu(0) < c - \varepsilon$, so in fact $F^{-1}([c-\varepsilon, c+\varepsilon])$ contains no critical points. Then by the weak Morse principle, $F^{-1}((-\infty, c-\varepsilon])$ is a deformation retract of $M^{c+\varepsilon} = F^{-1}((-\infty, c+\varepsilon])$.

Let $H := \overline{F^{-1}((-\infty, c-\varepsilon]) - M^{c-\varepsilon}}$, so that $F^{-1}((-\infty, c-\varepsilon]$ is $M^{c-\varepsilon} \cup H$, or $M^{c-\varepsilon}$ with the 'handle' H attached. Since $\xi \leq \varepsilon$ and $\eta = 0$ for all $q \in e^\lambda$, $F(q) \leq F(p) < c - \varepsilon$, since $\frac{\partial F}{\partial \xi} < 0$. On the other hand, $f(q) \geq c - \varepsilon$ for all $q \in e^\lambda$. We thus deduce that $e^\lambda \subseteq H$.

The final step of the proof consists in showing that $M^{c-\varepsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\varepsilon} \cup H$. Let r_t be the identity outside of U. We split the definition of r_t up into three cases.

Case 1. $\xi \leq \varepsilon$. Let $r_t(u^1, \cdots, u^n) := (u^1, \cdots, u^\lambda, tu^\lambda, \cdots, tu^n)$.

Case 2. $\varepsilon \leq \xi \leq \eta + \varepsilon$. Let $r_t(u^1, \cdots, u^n) := (u^1, \cdots, u^\lambda, s_t u^\lambda, \cdots, s_t u^n)$, where $s_t := t + (1 - t)\sqrt{(\xi - \varepsilon)/\eta}$.

Case 3. $\xi \geq \eta + \varepsilon$ (i.e., within $M^{c-\varepsilon}$). Let r_t be the identity.

The reader is invited to verify that the definitions of r_t agree at the endpoints of each case and that it is indeed the retraction we desire. □

Actually, based on a slightly different proof of Thm. 2.12, we may do away with the compactness assumption on sublevel sets in Thms. 2.13 and 2.14 above; see [10, Cor. 6.7].

We have the following important consequence of the principles just formulated.

Theorem 2.15. *Any smooth compact manifold M has the homotopy type of a finite CW-complex. Given a Morse function f on M, this CW-complex consists of one λ-cell for each critical point of f of index λ.*

2.4 The Morse Inequalities

This section will assume some knowledge of algebraic topology and/or homological algebra as covered in [6] or similar texts. Specifically, we will need relative homology and excision.

Definition 2.16. Let S be a function sending pairs of spaces to integers. If $X \supseteq Y \supseteq Z$, we say that S is *subadditive* if $S(X,Z) \leq S(X,Y) + S(Y,Z)$.

For instance, the ith *Betti number* $\beta_i(X,Y)$ of topological spaces $X \supseteq Y$ is the rank of the ith relative homology group $H_i(X,Y)$. Equivalently, $\beta_i(X,Y) := \dim_{\mathbb{R}} H_i(X,Y;\mathbb{R})$. β_i is subadditive; this can be verified by examining the long exact sequence

$$\cdots \to H_i(Y,Z) \to H_i(X,Z) \to H_i(X,Y) \to \cdots$$

Similarly, the *Euler characteristic* of $X \supseteq Y$, $\chi(X,Y) := \sum_i (-1)^i \beta_i(X,Y)$, is also subadditive. (In fact, it is additive.)

Lemma 2.17. *Suppose S is subadditive and $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n$ is an ascending chain of spaces. Then $S(X_n, X_0) \leq \sum_{i=1}^{n} S(X_i, X_{i-1})$. If S is additive, this is an equality.*

Proof. Proceed by induction on n. $n = 1$ is trivial and $n = 2$ is the definition of subadditivity. Suppose the statement holds for $n-1$. Then

$$\begin{aligned} S(X_n, X_0) &\leq S(X_n, X_{n-1}) + S(X_{n-1}, X_0) \\ &\leq S(X_n, X_{n-1}) + \sum_{i=1}^{n-1} S(X_i, X_{i-1}) \\ &= \sum_{i=1}^{n} S(X_i, X_{i-1}). \end{aligned}$$

If S is additive, simply replace all inequalities with equality. □

Let f be a Morse function on a compact smooth manifold M. Let $a_1 < \cdots < a_k$ such that M^{a_i} contains exactly i critical points of f, and $M^{a_k} = M$. By our lemma,

$$\begin{aligned} H_*(M^{a_i}, M^{a_{i-1}}) &= H_*(M^{a_{i-1}} \cup e^{\lambda_i}, M^{a_{i-1}}) \\ &= H_*(e^{\lambda_i}, \partial e^{\lambda_i}) \quad \text{by excision} \\ &= \mathbb{Z}^{\lambda_i}, \end{aligned}$$

where λ_i is the index of the critical point passing from $M^{a_{i-1}}$ to M^{a_i}.

Applying Lem. 2.17 to $\varnothing = M^{a_0} \subseteq M^{a_1} \subseteq \cdots \subseteq M^{a_n} = M$, $S = \beta_\lambda$, we have

$$\beta_\lambda(M) = \beta_\lambda(M, \varnothing) \leq \sum_{i=1}^{k} \beta_\lambda(M^{a_i}, M^{a_{i-1}}) = C_\lambda,$$

where C_λ denotes the number of critical points of index λ. Applying the lemma to the same sequence and $S = \chi$, we have

$$\chi(M) = \sum_{i=1}^{k} \chi(M^{a_i}, M^{a_{i-1}}) = \sum (-1)^i C_i.$$

We record what we have shown:

Proposition 2.18 (Weak Morse Inequalities)**.** *If C_λ is the number of critical points of index λ on M, then*

$$\beta_\lambda(M) \leq C_\lambda \quad \textit{and} \quad \chi(M) = \sum_{\lambda} (-1)^\lambda C_\lambda.$$

Lemma 2.19. *The function* $S_\lambda(X,Y) := \beta_\lambda(X,Y) - \beta_{\lambda-1}(X,Y) + \cdots \pm \beta_0(X,Y)$ *is subadditive.*

Proof. Given the exact sequence of vector spaces

$$\cdots \xrightarrow{\zeta} A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \cdots \to D \to 0,$$

we have that $\dim A = \operatorname{rank}\alpha + \dim\ker\alpha = \operatorname{rank}\alpha + \operatorname{rank}\zeta$. Thus,

$$\begin{aligned}\operatorname{rank}\zeta &= \dim A - \operatorname{rank}\alpha \\ &= \dim A - (\dim B - \operatorname{rank}\beta) \\ &= \dim A - \dim B + \dim C - \operatorname{rank}\gamma \\ &= \cdots \\ &= \dim A - \dim B + \dim C - \cdots \pm \dim D \geq 0.\end{aligned}$$

Now apply this to the map $\partial : H_{\lambda+1}(X,Y) \to H_\lambda(Y,Z)$ in the relative homology exact sequence of $X \supseteq Y \supseteq Z$:

$$\operatorname{rank}\partial = \beta_\lambda(Y,Z) - \beta_\lambda(X,Z) + \beta_\lambda(X,Y) - \beta_{\lambda-1}(Y,Z) + \beta_{\lambda-1}(X,Z) - \beta_{\lambda-1}(X,Y) + \cdots \geq 0.$$

Therefore $S_\lambda(Y,Z) - S_\lambda(X,Z) + S_\lambda(X,Y) \geq 0$, which is the inequality desired. □

This proves the following upgrade:

Theorem 2.20 (Morse Inequalities)**.** *If* C_λ *is the number of critical points of index* λ *on* M*, then*

$$\beta_\lambda(M) - \beta_{\lambda-1}(M) + \cdots \pm \beta_0(M) \leq C_\lambda - C_{\lambda-1} + \cdots \pm C_0.$$

Proof.

$$\begin{aligned}\beta_\lambda(M) - \beta_{\lambda-1}(M) + \cdots \pm \beta_0(M) &= S_\lambda(M) \\ &\leq \sum_{i=1}^{k} S_\lambda(M^{a_i}, M^{a_{i-1}}) \\ &= C_\lambda - C_{\lambda-1} + \cdots \pm C_0.\end{aligned}$$

□

Let us now introduce some common terminology.

Definition 2.21.

i) The *Poincaré polynomial* $P_X(t)$ of a space X with finitely generated homology (e.g., a compact topological manifold) is the generating function for its Betti numbers, that is:

$$P_X(t) := \sum_{i\geq 0} \beta_i(X) t^i.$$

ii) The *Morse polynomial* $\mathcal{M}_f(t)$ of a Morse function f with finitely many critical points is the generating function for C_λ:

$$\mathcal{M}_f(t) := \sum_{p\in \mathbf{Cr}_f} t^{\lambda(f,p)} = \sum_{\lambda\geq 0} C_\lambda t^\lambda.$$

We can formulate the Morse inequalities in terms of these gadgets as well:

Corollary 2.22 (Abstract Morse Inequalities)**.** *Let* f *be a Morse function on* M*. There exists a polynomial* $Q(t) \in \mathbb{Z}[t]$ *with nonnegative coefficients such that*

$$\mathcal{M}_f(t) - P_M(t) = (1+t)Q(t).$$

Proof. Set $Q(t) := \sum_i q_i t^i$ and $c_i := C_i - \beta_i(M)$, so that

$$\mathcal{M}_f(t) - P_M(t) = \sum_i c_i t^i = q_0 + \sum_i (q_{i-1} + q_i) t^i.$$

Then

$$\begin{aligned} q_0 &= c_0 \\ q_1 &= c_1 - c_0 \\ &\vdots \\ q_i &= c_i - c_{i-1} + c_{i-2} - \cdots \pm c_0 \\ &\vdots \end{aligned}$$

The statement follows directly from Thm. 2.20. □

Definition 2.23. Let f be a Morse function on M. If the quantities in the Morse inequalities for f are in fact equal, then f is called a *perfect Morse function*. Equivalently, $\mathcal{M}_f(t) = P_M(t)$.

3 Applications

We leave the reader with a few tantalising applications of Morse theory.

3.1 A Theorem of Reeb

Proposition 3.1 (Reeb)**.** *If M is a compact manifold and f is a Morse function with exactly two critical points, then M is homeomorphic to a sphere.*

Proof. Let $n := \dim M$. The critical points p and q must be the minimum and maximum, respectively. Without loss of generality, we may assume that $f(p) = 0$ and $f(q) = 1$. Then $M^\varepsilon = f^{-1}([0, \varepsilon])$ and $f^{-1}([1-\varepsilon, 1])$ are closed n-discs by the Morse lemma, and $M^{1-\varepsilon}$ is homeomorphic to M^ε by Thm. 2.13. Since $M = M^{1-\varepsilon} \cup f^{-1}([1-\varepsilon, 1])$ glued along their common boundary, we conclude that $M \cong S^n$. □

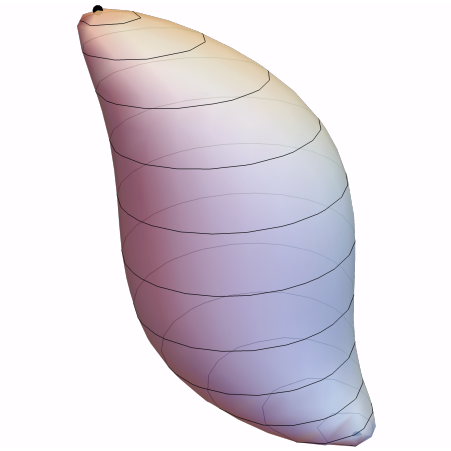


Figure 3

3.2 The Complex Grassmannian

Morse–Smale Dynamics

In the interest of brevity, we quote but do not prove the following results on *Morse–Smale dynamics* from [14, §2.4].

Suppose $f : M \to \mathbb{R}$ is a Morse function on the compact manifold M and ∇f is the gradient of f. The *descending flow* Φ_t is the flow on M determined by $-\nabla f$.

Proposition 3.2. *For every point $p \in M$, the limits*

$$\Phi_{\pm\infty}(p) := \lim_{t \to \pm\infty} \Phi_t(p)$$

exist and are critical points of f.

Define the *stable manifold* W_p^+ (respectively the *unstable manifold* W_p^-) *of the critical point* p as

$$W_p^{\pm} := \Phi_{\pm\infty}^{-1}(p).$$

Proposition 3.3. *Let $m := \dim M$ and $\lambda := \lambda(f, p)$. Then W_p^- is a smooth manifold homeomorphic to $\mathbb{R}^\lambda$, whereas W_p^+ is a smooth manifold homeomorphic to $\mathbb{R}^{m-\lambda}$.*

Complex Grassmannians

Recall that the *complex Grassmannian* $G_k(\mathbb{C}^n)$ is a smooth manifold of complex dimension $k(n-k)$ consisting of all k-dimensional $\mathbb{C}$-linear subspaces of $\mathbb{C}^n$. Given a Hermitian inner product on $\mathbb{C}^n$ (say, the standard one), there is a diffeomorphism $G_k(\mathbb{C}^n) \cong G_{n-k}(\mathbb{C}^n)$ which maps each k-dimensional subspace to its orthogonal complement in $\mathbb{C}^n$.

Let $P_{k,n}(t)$ be the Poincaré polynomial of $G_k(\mathbb{C}^n)$. We will compute $P_{k,n}(t)$ using Morse theory.

Take the standard Hermitian inner product on $\mathbb{C}^n$, $\langle z, w\rangle := \sum_{j=1}^{\infty} z^j \overline{w^j}$. Consider the orthogonal projection $\pi_V : \mathbb{C}^n \to \mathbb{C}^n$ onto a k-dimensional linear subspace V of $\mathbb{C}^n$; $i\pi_V$ is then a skew-adjoint operator, hence a skew-adjoint matrix belonging to the real Lie algebra $\mathfrak{u}(n) := \{A \in M_n(\mathbb{C}) : A + A^* = 0\}$ of the unitary group $U(n)$. It is not difficult to see that the *adjoint action* of $U(n)$ on $\mathfrak{u}(n)$ ($\mathrm{Ad}_g(A) := gAg^{-1}$) is transitive on this embedding $G_k(\mathbb{C}^n) \hookrightarrow \mathfrak{u}(n)$ with stabiliser $U(k) \times U(n-k)$; in fact, $G_k(\mathbb{C}^n)$ is an *adjoint orbit* of the block matrix

$$\begin{pmatrix} iI_k & 0 \\ 0 & 0 \end{pmatrix}.$$

To proceed, we will need a fact from Lie theory, namely:

Fact. Every adjoint orbit intersects a *Cartan subalgebra* (a maximal abelian subalgebra) of a Lie algebra nontrivially, in an orbit of the Weyl group of the Lie group.

For $\mathfrak{u}(n)$, we can take the Cartan subalgebra to be

$$\mathfrak{h} := \left\{ A : A = \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_n \end{pmatrix} \right\}.$$

We can interpret this as the familiar spectral theorem from linear algebra: every skew-adjoint is unitarily diagonalisable. In this case, the intersection of $\mathfrak{h}$ with $G_k(\mathbb{C}^n)$ is an orbit of S_n, which permutes the n coordinate axes.

The Lie algebra $\mathfrak{u}(n)$ has an inner product given by

$$\langle A, B\rangle := \mathrm{tr}(AB^*) = -\mathrm{tr}(AB),$$

which is invariant under the adjoint action. Let $Z \in \mathfrak{h}$ with j^{th} diagonal entry ia_j, where $a_1 > \cdots > a_n \geq 0$. Define $f : G_k(\mathbb{C}^n) \to \mathbb{R}$ as $f(V) := \langle i\pi_V, Z\rangle$.

We claim that the critical points of this function are precisely the $\binom{n}{k}$ subspaces $V_I := \mathrm{span}_{\mathbb{C}}\{e_i\}_{i \in I}$, where $I := (i_1, \cdots, i_k), 1 \leq i_1 < \cdots < i_k \leq n$ is a multi-index.

Recall that $T_V G_k(\mathbb{C}^n) \cong \mathrm{Hom}_{\mathbb{C}}(V, V^\perp)$. Observe that the differential of f vanishes precisely when $ZV \subseteq V$. But Z has distinct eigenvalues, so Z-invariant subspaces are the spans of any number of its eigenvectors. This proves our claim.

It can be verified that the gradient flow $-\nabla f$ has an integral curve through V given by

$$\gamma(t) := e^{-itZ}V = \begin{pmatrix} e^{a_1 t} & & \\ & \ddots & \\ & & e^{a_n t} \end{pmatrix} V.$$

Putting V in reduced column echelon form (v_{ij}) (thus representing the same element of $G_k(\mathbb{C}^n)$), we see that

$$\lim_{t\to-\infty} e^{-itZ}V = \lim_{t\to-\infty} e^{-itZ} \begin{pmatrix} 1 & & \\ * & \ddots & \\ * & * & 1 \\ | & | & | \end{pmatrix}$$

For this limit to go to V_I, i.e., for the i_l^{th} row to go to $(0, \cdots, 1, \cdots, 0)$, with 1 in the lth column and 0's elsewhere, note that all entries v_{ij} with $j > l$ must be 0 to begin with for $i_l \le i < i_{l+1}$. Additionally, the i_l^{th} row itself must have all entries $v_{i_l j}$ equal to 0 for $j < l$. The rest of the entries are free, since $e^{(a_i - a_k)t}$ goes to 0 as $t \to -\infty$, and we count

$$(n - i_1 - (k-1)) + (n - i_2 - (k-2)) + \cdots + (n - i_k - (k-k)) = \boxed{k(n-k) + \frac{k(k+1)}{2} - \sum_{j=1}^{k} i_j} =: d$$

free entries.

(As an example, take $n = 7, k = 3$, and $I = (2, 4, 6)$. Then $V_I = \begin{pmatrix} 0&0&0 \\ 1&0&0 \\ 0&0&0 \\ 0&1&0 \\ 0&0&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix}$. $V \in W_{V_I}^-$ must therefore be of the form

$$V = \begin{pmatrix} 0&0&0 \\ 1&0&0 \\ *&0&0 \\ 0&1&0 \\ *&*&0 \\ 0&0&1 \\ *&*&* \end{pmatrix}.$$

Note that there are 6 free entries, which matches our formula.)

Thus, the unstable manifold $W_{V_I}^-$ is homeomorphic to $\mathbb{C}^d \cong \mathbb{R}^{\lambda(f, V_I)}$, and we obtain

$$\lambda(f, V_I) = 2\left(k(n-k) + \frac{k(k+1)}{2} - \sum_{j=1}^{k} i_j\right).$$

It is then a matter of some arithmetic to deduce that the Morse polynomial of $G_k(\mathbb{C}^n)$ has the particularly nice form

$$\mathcal{M}_f(t) = \frac{\prod_{i=1}^{n}(1 - t^{2i})}{\prod_{i=1}^{k}(1 - t^{2i}) \prod_{i=1}^{n-k}(1 - t^{2i})} = \binom{n}{k}_{t^2} \in \mathbb{Z}[t^2],$$

where the symbol on the RHS is known as the *Gaussian binomial coefficient*.

Since there is no polynomial $P_M(t)$ which we could subtract from $\mathcal{M}_f(t)$ to get a $(1+t)Q(t)$, as in the Morse inequalities (Cor. 2.22), we conclude f is a perfect Morse function, and $P_{k,n}(t) = \mathcal{M}_f(t)$.

In particular, $\mathbb{C}P^n = G_1(\mathbb{C}^n)$ has Poincaré polynomial $(1 - t^{2n})/(1 - t^2) = 1 + t^2 + \cdots + t^{2n}$, so has the homotopy type of a CW complex of the form $e^0 \cup e^2 \cup \cdots \cup e^{2n}$. We can thus calculate its integral homology:

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Computing its cohomology is, of course, a straightforward application of Poincaré duality.

Something to note: as in the proof of the weak Morse principle (Thm. 2.13), here the gradient flow of a Morse function f enters explicitly through Morse–Smale dynamics. Indeed, especially in modern approaches to Morse theory, most notably those of Edward Witten and Andreas Floer, the gradient flow and its flow lines (i.e., integral curves) have come to play a central role.

3.3 The Lefschetz Hyperplane Theorem

The Lefschetz hyperplane theorem was originally proven by Lefschetz in 1924 [9]. Here we present the approach put forward by Andreotti and Fraenkel [1], inspired by an earlier Morse-theoretic proof by Thom.

Theorem 3.4 (Andreotti–Fraenkel)**.** *Let M be a complex-analytic manifold of complex dimension m, biholomorphically embedded as a closed subset of $\mathbb{C}^n$. Then M has the homotopy type of an m-dimensional CW complex.*

Such an M is called a *Stein manifold*.

Corollary 3.5. *In particular, if $V \subseteq \mathbb{C}^n$ is a nonsingular affine variety with real dimension $2m$, then*

$$H_i(V) = 0 \text{ for } i > m.$$

Before we pick up the proof of Thm. 3.4, let us review some definitions from complex differential geometry.

Suppose M is a complex manifold of complex dimension m. The real tangent bundle TM comes equipped with a natural automorphism $J : TM \to TM$ such that $J^2 = -1$, known as the *almost complex structure*. If (z^j) are complex coordinates on M with $z^j = x^j + iy^j$, then $J\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}$ and $J\frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}$.

J extends by linearity to $J \otimes \mathbb{C}$ on the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$; $J^2 = -1$ shows that $\pm i$ are the only eigenvalues of J on the complexified TM. Setting $TM^{1,0} := \ker(i - J \otimes \mathbb{C}), TM^{0,1} := \ker(i + J \otimes \mathbb{C})$, we have that $TM \otimes_{\mathbb{R}} \mathbb{C} = TM^{1,0} \oplus TM^{0,1}$. With coordinates (z^j) as before, $TM^{1,0}$ is spanned by $\frac{\partial}{\partial z^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right)$; whereas $TM^{0,1}$ is spanned by $\frac{\partial}{\partial \overline{z}^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right)$.

J induces its transpose $J^{\mathrm{T}} : T^*M \to T^*M$ which again extends to $J^{\mathrm{T}} \otimes \mathbb{C}$ on the complexified cotangent bundle. There is once again the decomposition $T^*M \otimes_{\mathbb{R}} \mathbb{C} = T^*M^{1,0} \oplus T^*M^{0,1}$, locally spanned by $dz^j := dx^j + i dy^j$ and $d\overline{z}^j := dx^j - i dy^j$, respectively.

This induces a decomposition

$$\Lambda^r(T^*M \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{p+q=r} \Lambda^{p,q}(T^*M),$$

where $\Lambda^{p,q}(T^*M) = \Lambda^p(T^*M^{1,0}) \otimes_{\mathbb{C}} \Lambda^q(T^*M^{0,1})$, with obvious bases in terms of wedge products of dz^j's and $d\overline{z}^j$'s, respectively. The space of smooth sections of $\Lambda^{p,q}(T^*M)$ is denoted $\Omega^{p,q}(M)$; its elements are aptly named *(p,q)-forms*.

There is, as in the real case, the *exterior derivative* of a (p,q)-form α: $d\alpha = (d\alpha)^{p+1,q} + (d\alpha)^{p,q+1}$. Denote the resulting $(p+1,q)$- and $(p,q+1)$-forms by $\partial\alpha := (d\alpha)^{p+1,q}$, $\overline{\partial}\alpha := (d\alpha)^{p,q+1}$, respectively.

We now state two lemmas, whose proofs are a matter of tracing the definitions just laid out.

Lemma 3.6 (Complex Hessian)**.** *Suppose $f : M \to \mathbb{R}$ is a smooth function on the complex manifold M and p is a critical point of f. Let $H := H_{f,p}$ be the Hessian of f at p. Define the* complex Hessian *or* Levi form *of f at p to be the $\mathbb{R}$-bilinear form*

$$L_{f,p} : T_pM \times T_pM \to \mathbb{R};\ L_{f,p}(u,v) := H(u,v) + H(Ju, Jv).$$

Then we have the following identity:

$$L_{f,p}(u,v) = i\partial\overline{\partial} f(u, Jv).$$

Lemma 3.7 (Pseudoconvexity)**.** *Let the* distance-squared function $f_{\mathbf{a}} : \mathbb{C}^n \to \mathbb{R}$ *be defined as*

$$f_{\mathbf{a}}(\mathbf{z}) := \frac{1}{2}|\mathbf{z} - \mathbf{a}|^2 = \frac{1}{2}(|z^1 - a^1|^2 + \cdots + |z^n - a^n|^2).$$

Then for all $q \in \mathbb{C}^n$ *and each real tangent vector* $v \in T_q\mathbb{C}^n$*, we have*

$$i(\partial\overline{\partial} f_{\mathbf{a}})_q(v, Jv) = |v|^2.$$

Let us now proceed with the proof of the main result.

Proof of Thm. 3.4. Let $M \hookrightarrow \mathbb{C}^n$ be a Stein manifold of complex dimension m, and let $f := f_{\mathbf{a}}$ be as in Lem. 3.7. By choosing a generic point $\mathbf{a} \in \mathbb{C}^n$, the restriction of f onto M is a Morse function on M, as per the parametric transversality theorem [8, Thm. 6.35]. Suppose p is a critical point of $f|_M$ and let H be the (real) Hessian of $f|_M$ at p.

We wish to prove that the index of p is at most m, so that the theorem follows from the strong Morse principle (Thm. 2.14). Unrolling the definition of the index of a critical point, we wish to prove that for any $\mathbb{R}$-linear subspace $W \subseteq T_pM$ such that H is negative-definite on W, $\dim_{\mathbb{R}} W \leq m$.

Suppose for the sake of contradiction that $W \cap JW \neq 0$, so there exists $v \in W - \{0\}$ such that $Jv \in W$. Then

$$\begin{aligned} L_{f,p}(v, v) &= H(v, v) + H(Jv, Jv) \\ &< 0 + 0 = 0. \end{aligned}$$

By Lem. 3.6, we have that

$$L_{f,p}(v, v) = i(\partial\overline{\partial} f|_M)_p(v, Jv) = i(\partial\overline{\partial} f)_p(v, Jv) < 0.$$

But by the pseudoconvexity lemma,

$$i(\partial\overline{\partial} f)_p(v, Jv) = |v|^2.$$

This implies that $|v|^2 < 0$; which is absurd. Hence, $W \cap JW = 0$, and therefore

$$2m = \dim_{\mathbb{R}} T_pM \geq \dim_{\mathbb{R}} W + \dim_{\mathbb{R}} JW = 2 \dim_{\mathbb{R}} W. \qquad \square$$

Now suppose $V \subseteq \mathbb{C}P^n$ is a nonsingular projective variety, i.e., the zero locus of homogeneous polynomials $Q_1, \cdots, Q_r \in \mathbb{C}[z^0, \cdots, z^n]$. Consider a hyperplane $\pi \subseteq \mathbb{C}P^n$, which we may assume to be $\pi := \{[z^0 : z^1 : \cdots : z^n] : z^0 = 0\}$. Let $V_\pi := V \cap \pi$ and $M := V - V_\pi$. Then M is a nonsingular affine subvariety of $\mathbb{C}^n$, being the zero locus of polynomials $P_j(z^1, \cdots, z^n) := Q_j(1, z^1, \cdots, z^n)$ for $1 \leq j \leq r$, in fact a Stein manifold.

By Cor. 3.5, we have that $H_{m+j}(M) = 0$ for all $j > 0$. But by Poincaré–Lefschetz duality [6, Thm. 3.43], we have an isomorphism $H_k(V - V_\pi) \cong H^{2m-k}(V, V_\pi; \mathbb{Z})$. Thus, $H^{m-j}(V, V_\pi; \mathbb{Z}) = 0$ for $j > 0$. By the long exact sequence of the pair (V, V_π),

$$\cdots \to H^{m-j}(V, V_\pi; \mathbb{Z}) \to H^{m-j}(V; \mathbb{Z}) \to H^{m-j}(V_\pi; \mathbb{Z}) \xrightarrow{\delta} H^{m-j+1}(V, V_\pi; \mathbb{Z}) \to \cdots,$$

we have that $H^{m-j}(V; \mathbb{Z}) \to H^{m-j}(V_\pi; \mathbb{Z})$ is an isomorphism for $j > 1 \Leftrightarrow m - j < \frac{1}{2}\dim_{\mathbb{R}} V_\pi$, and is injective for $j = 1 \Leftrightarrow m - j = \frac{1}{2}\dim_{\mathbb{R}} V_\pi$. We thus obtain the Lefschetz theorem on hyperplane sections in its original form:

Theorem 3.8 (Lefschetz Hyperplane Theorem)**.** *Suppose* V *is a nonsingular projective variety and* V_π *is a* hyperplane section *of* V *(the intersection of* V *with a hyperplane of* $\mathbb{C}P^n$*). Then the natural map*

$$H^k(V; \mathbb{Z}) \to H^k(V_\pi; \mathbb{Z})$$

is an isomorphism if $k < \frac{1}{2}\dim_{\mathbb{R}} V_\pi$*, and is injective if* $k = \frac{1}{2}\dim_{\mathbb{R}} V_\pi$*.*

Another more geometric approach is given in [10, §I.7] involving *focal point analysis* of the distance-squared function, with the added benefit of allowing us to extend Thm. 3.8 to homotopy groups $\pi_k(V) \to \pi_k(V_\pi)$.

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