Part of the proof of Dirichlet's theorem

Jake Lai

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Dirichlet's theorem states that there are infinitely many primes in the arithmetic progression $aq+b$, where $\gcd(q,b)=1$. In fact, Dirichlet showed an even stronger result:

$$\sum_{p \equiv b \pmod{q}} \frac{1}{p} = \infty,$$

where the sum is taken over all primes p. It turns out that this reduces to the fact that the L-function $L(s,\chi) = \sum \frac{\chi(n)}{n^s}$ is nonzero at $s=1$.

The following exposition of Dirichlet's theorem is largely adapted from Stein & Shakarchi's *Fourier Analysis*. Here, $\mathbb{Z}$ is the set of integers, $\mathbb{C}$ the set of complex numbers, and $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ the unit circle.

1 Basic properties

Definition 1.1. A *Dirichlet character* $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to S^1$ is a completely multiplicative function from the group of (equivalence classes of) integers coprime to q under multiplication to the circle group S^1 (i.e., a group homomorphism). That is, $\chi(mn) = \chi(m)\chi(n)$ for all m, n.

We can extend the domain of χ to all of $\mathbb{Z}$ by letting $\chi(n+q) = \chi(n)$ whenever n is coprime to q, and $\chi(n) = 0$ otherwise.

In particular, call the character

$$\chi_0(n) := \begin{cases} 1 \text{ if } \gcd(n,q) = 1 \\ 0 \text{ otherwise} \end{cases}$$

the *trivial character*.

Fixing q, we may speak of the set of all Dirichlet characters (which has a group structure).

Definition 1.2. A *L-function* is defined by the series

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

for any Dirichlet character χ.

Proposition 1.3 (Euler product formula)**.** *L-functions satisfy the following formula:*

$$L(s,\chi) = \prod_p \left(\frac{1}{1-\chi(p)p^{-s}} \right),$$

where the product is taken over all primes p.

To see this, simply multiply the series expansion of the LHS by successive factors of $(1-\chi(p)p^{-s})$; this goes to 1, making use of Erastothenes's sieve and the complete multiplicativity of χ.

In particular, if $q = p_1^{a_1} p_2^{a_2} \cdots p_N^{a_N}$, then

$$L(s,\chi_0) = (1-p_1^{-s}) \cdots (1-p_N^{-s}) \cdot \zeta(s),$$

where ζ is the Riemann zeta function.

2 Reduction to nonvanishing of $L(s,\chi)$

Remarkably, due to abstract Fourier analysis (aka abstract harmonic analysis), we have the *orthogonality relations*:

$$\frac{1}{\varphi(q)} \sum_\chi \overline{\chi(m)}\chi(n) = \delta_{m,n} := \begin{cases} 1 \text{ if } m \equiv n \pmod q \\ 0 \text{ otherwise,} \end{cases}$$

where φ is Euler's totient function (in fact, the cardinality of $(\mathbb{Z}/q\mathbb{Z})^\times$) and the sum is taken over all Dirichlet characters χ. Thus,

$$\begin{aligned} \sum_{p \equiv b \pmod q} \frac{1}{p^s} &= \sum_p \frac{\delta_{b,p}}{p^s} \\ &= \frac{1}{\varphi(q)} \sum_\chi \overline{\chi(b)} \sum_p \frac{\chi(p)}{p^s} \\ &= \frac{1}{\varphi(q)} \left[\sum_p \frac{\chi_0(p)}{p^s} + \sum_{\chi \neq \chi_0} \overline{\chi(b)} \sum_p \frac{\chi(p)}{p^s} \right]. \end{aligned}$$

There are only finite many primes p such that $\chi_0(p) = 0$, so $\sum_p \frac{\chi_0(p)}{p^s}$ diverges as s tends to 1. This leave us to prove that $L(s,\chi)$ is bounded as $s \to 1^+$ for nontrivial χ.

If $L(1, \chi)$ is finite and nonzero, then since

$$\begin{aligned}
\log L(s, \chi) &= -\sum_p \log(1 - \chi(p)p^{-s}) \\
&= -\sum_p \left[-\chi(p)p^{-s} + O(p^{-2s})\right] \\
&= \sum_p \frac{\chi(p)}{p^s} + O(1),
\end{aligned}$$

$\log L(s, \chi)$ is bounded as $s \to 1^+$; therefore the sum $\sum_p \frac{\chi(p)}{p^s}$ remains bounded as well. It turns out that $L(s, \chi)$ is continuous at $s = 1$, so we need only show that $L(1, \chi)$ does not vanish.

3 Proof of nonvanishing

Theorem 3.1. *If $\chi \neq \chi_0$, then $L(1, \chi) \neq 0$.*

We split into two cases: where χ takes on nonreal values, i.e., $\chi \neq \overline{\chi}$; and where χ takes on real values only, i.e., $\chi(n) \in \{0, \pm 1\}$.

3.1 Case 1: χ nonreal

We use the following two lemmas.

Lemma 3.2.

$$\prod_\chi L(s, \chi) \geq 1,$$

where the product is taken over all Dirichlet characters χ.

Proof. Observe that

$$\begin{aligned}
\prod_\chi L(s, \chi) &= \exp\left[\sum_\chi \log L(s, \chi)\right] \\
&= \exp\left[\sum_\chi \sum_p \log\left(\frac{1}{1 - \chi(p)p^{-s}}\right)\right] \\
&= \exp\left[\sum_\chi \sum_p \sum_{k=1}^{\infty} \frac{\chi(p^k)}{p^{ks}}\right] \\
&= \exp\left[\sum_p \sum_{k=1}^{\infty} \frac{1}{p^{ks}} \sum_\chi \chi(p^k)\right].
\end{aligned}$$

If we take $m = 1$ and $n = p^k$, we get $\sum_\chi \chi(p^k) = \varphi(q)\delta_{1,p^k}$ from the orthogonality relations. Hence,

$$\prod_\chi L(s,\chi) = \exp\left[\sum_p \sum_{k=1}^\infty \frac{\varphi(q)\delta_{1,p^k}}{p^{ks}}\right].$$

The last expression is greater than or equal 1 because the each term in the exponential is nonnegative. □

Lemma 3.3. *The following hold for $s \geq 1$:*

1. *If $L(1,\chi) = 0$, then $L(1,\overline{\chi}) = 0$.*
2. *If $\chi \neq \chi_0$, and $L(1,\chi) = 0$, then $L(s,\chi) \leq C|s-1|$.*
3. *$L(s,\chi_0) \leq C/|s-1|$.*

We can now prove Theorem 3.1 on the nonvanishing of L-functions of nonreal Dirichlet characters at $s = 1$.

Proof. Let χ be a nonreal (thus, nontrivial) Dirichlet character. Suppose $L(1,\chi) = 0$. Then there are two distinct terms, $L(s,\chi)$ and $L(s,\overline{\chi})$, that contribute to $\prod_\chi L(s,\chi)$; these vanish in $O(|s-1|)$ as $s \to 1^+$, so together vanish in $O(|s-1|^2)$. Only the term $L(s,\chi_0)$ tends to infinity like $O(1/|s-1|)$, which is insufficient to prevent $\prod_\chi L(s,\chi)$ from vanishing. This contradicts Lemma 3.2. □

3.2 Case 2: χ real

(Under construction.)