Diagnostician

Jake Lai

(You need a minimum of high school/JC/IB/etc. calculus and all its prerequisites. Problems roughly go from less to more 'applied', whatever that means.)

Preliminaries:

':=' means 'is defined as'.

Sets are collections of arbitrary things we call *elements*.

Write $x \in S$ to indicate an element x is contained in S.

Write $T \subseteq S$ to denote that T is a *subset* of S, i.e., '$x \in T$ implies $x \in S$'.

The *natural numbers* is the set $\mathbb{N} := \{1, 2, 3, \cdots\}$.

The familiar *real number line* is usually written $\mathbb{R}$, whose elements include $-\frac{1}{3}$, $\sqrt{2}$, and π^e.

Functions (written $f : S \to T$) are rules that take a given element $x \in S$ and return an element $y \in T$; more familiarly, $y = f(x)$.

1 Pure

1.1 Logic

How might we compare the sizes of infinite sets? It makes sense that two sets have the same size if there is a function $f : S \to T$ such that f has an '*inverse*' $f^{-1} : T \to S$ satisfying $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$ for all $x \in S, y \in T$. Functions with inverses are called *bijections*. The existence of a bijection is, in fact, the definition of two sets having equal size (or *cardinality*).

Problem. A bijection between $\mathbb{N}$ and another set S is the same as finding a 'counting' procedure to list out all elements of S one by one. Find a bijection between $\mathbb{N}$ and $\mathbb{Z}$ (the set of all integers). Find a bijection between $\mathbb{N}$ and $\mathbb{Q}_+$ (the set of all positive fractions) by laying the elements of $\mathbb{Q}_+$ out on a 2D grid.

(Hint: It doesn't work to count the positive rational numbers going 'horizontally', nor does it work to go 'vertically' either. The counting procedure needs to cover each and every positive rational number. [Sub-hint: You have to 'snake' across the 2D grid of positive rational numbers — but how?])

We can conclude that $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}_+| = |\mathbb{Q}|$, where $|S|$ denotes the size of the set S.

Bonus. (*Cantor diagonalisation*) We prove that no bijection exists between $\mathbb{N}$ and $R := (0, 1)$ (the set of real numbers $0 < x < 1$). Identify the elements of the latter with the decimal expansion $x = 0.x_1x_2x_3\cdots$. First assume that there is such a bijection, so that we may list the elements of R sequentially:

$$\begin{array}{lllll} 0\,. & \boxed{x_1^{(1)}} & x_2^{(1)} & x_3^{(1)} & \cdots \\ 0\,. & x_1^{(2)} & \boxed{x_2^{(2)}} & x_3^{(2)} & \cdots \\ 0\,. & x_1^{(3)} & x_2^{(3)} & \boxed{x_3^{(3)}} & \cdots \\ & & \vdots & & \end{array}$$

By our assumption, this list should include every element of R. However, I can produce an element of R outside the list: $0.y_1y_2y_3\cdots$, by defining $y_n := \begin{cases} 1 \text{ if } x_n^{(n)} = 0 \\ 0 \text{ if } x_n^{(n)} \neq 0 \end{cases}$. Since our construction differs from every element in the list by at least one digit, it lies outside the list. Thus, our assumption must be false, and there is no bijection $\mathbb{N} \leftrightarrow R$. Can you produce a bijection between R and $\mathbb{R}$? What does this say about $|\mathbb{N}|$ and $|\mathbb{R}|$?

1.2 Algebra/Combinatorics

A *group* is a set G together equipped with a *binary operator* (read: function) $\mu : G \times G \to G$. If $a, b \in G$, we will be lazy and write $a \cdot b$ or ab when we really mean $\mu(a, b)$. Groups (G, μ) are required to follow certain defining rules (*axioms*):

1. (*Associativity*) For all $a, b, c \in G$, $(ab)c = a(bc)$.
2. (*Identity*) There exists some element $e \in G$ (the '*identity*') such that, for all $a \in G$, $ae = a = ea$.
3. (*Inverses*) For all $a \in G$, there exists $b \in G$ (commonly denoted a^{-1} and called the '*inverse*') such that $ab = ba = e$.

(Note that groups need not be commutative, i.e., generally $ab \neq ba$.)

In order to even state the axiom on inverses, we need to prove that the identity element of a given group G is unique.

Problem. Let $e \in G$ be an identity. Prove that identities are unique.

(Hint: You just need to show that if $e_1, e_2 \in G$ are identities, then $e_1 = e_2$. [Sub-hint: You have been given two elements. What can you do with them?])

Bonus. If G is finite, prove that there is a positive integer $m \in \mathbb{N}$ such that for all $a \in G$, $a^m = e$.

(Hint: Consider the sequence

$$a, a^2, a^3, \cdots, a^m, \cdots, a^n, \cdots .$$

[Sub-hint: Apply the *pigeonhole principle*. Look it up if you don't know what that is.])

1.3 Analysis

A *sequence* is a list of real numbers $x_1, x_2, x_3, \cdots \in \mathbb{R}$, which we will commonly write $(x_i)_{i \in \mathbb{N}}$ or simply (x_i). A subsequence $(x_{i_k})_{k \in \mathbb{N}}$ is a sequence where $1 \leq i_1 < i_2 < i_3 < \cdots$ are all integers. We would like to show that all sequences (x_i) have a infinite subsequence (x_{i_k}) that is either *nondecreasing* (if $i < j$, then $x_i \leq x_j$) or *nonincreasing* (if $i < j$, then $x_i \geq x_j$).

Observe that every (x_i) has 'peaks' at some integer n, where $x_m \leq x_n$ for every $m > n$.

Problem. [Case 1] If there are infinitely many peaks at $i_1, i_2, i_3, \cdots$, what be said about the infinite subsequence (x_{i_k}) of such peaks? [Case 2] On the other hand, if there are finitely many peaks $i_1, i_2, \cdots, i_N$, how should we construct (read: give a procedure for choosing elements of) a subsequence (x_{j_k}), where $j_1 = i_N + 1$?

(Hint [Case 2]: j_1 is not a peak, so what is an option for the choice of j_2? Continuing, j_k is not a peak, so what should we choose for j_{k+1}?)

Bonus. (*Bolzano–Weierstrass*) A sequence (x_i) is *bounded* if there exist $a, b \in \mathbb{R}$ such that for all $i \in \mathbb{N}$, $a \leq x_i \leq b$ (a/b are called lower/upper bounds respectively). If you know that a bounded nondecreasing/nonincreasing sequence (x_i) gets arbitrarily close ('*converges*') to its upper/lower bound respectively, then do all bounded sequences have a converging subsequence?

2 Applied

2.1 Analysis

See 1.3 above.

2.2 Probability

WIP. Probably Markov chain.

2.3 Numerics

WIP. Probably Lagrange interpolation.

3 Statistics

3.1 Analysis

See 1.3 above.

3.2 Probability

See 2.2 above.

3.3 Statistics

WIP. Probably unbiased estimator of σ^2.