The following exposition of Loewner's torus inequality is based on Chapters 3-6 of the monograph *Systolic Geometry and Topology* by Mikhail Katz.

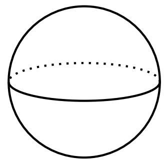
1 Geometric preliminaries

Def. By a (2-dimensional) closed *Riemannian manifold*, we mean a compact subset $\Sigma \subset \mathbb{R}^n$ whose every point has an open neighbourhood diffeomorphic to $\mathbb{R}^2$.

One uses the diffeomorphisms as a parametrisation to calculate, for example, the coefficients g_{ij} of the first fundamental form I. Denote by the manifold by $(\Sigma, \mathcal{G})$, where $\mathcal{G}$ is referred to as the 'metric'. Σ captures the topology. $\mathcal{G}$ captures the geometry.

For the intrinsic geometry of a Riemannian manifold, the immersion in $\mathbb{R}^n$ is irrelevant. In particular, manifolds such as *flat tori* are difficult to embed transparently.

Example: $S^2 \subset \mathbb{R}^3$. Letting $f(x,y) = \sqrt{1-x^2-y^2}$, $(u_1,u_2) \mapsto (u_1,u_2,f(u_1,u_2))$ is one chart. (We need six such charts.)



For vectors, we work with the dual bases $\frac{\partial}{\partial u_j}$ for vectors and du_i for covectors, such that

$$du_i\left(\frac{\partial}{\partial u_j}\right) = \delta_{i,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \quad \text{(the Kronecker delta)}$$

We can thus write the first fundamental form as follows:

$$\mathcal{G} = \sum_{i,j} g_{ij}(u_1,u_2)\, du_i du_j.$$

(I will ignore details re: differential forms, tensor algebra, etc., and also take for granted that every smooth manifold can be endowed with a Riemannian metric structure.)

Example: If $g_{ij} = \delta_{i,j}$, i.e., $\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $\mathcal{G} = (du_1)^2 + (du_2)^2$.

Def. Define the *area* of $(\Sigma, \mathcal{G})$ as

$$\text{area}(\Sigma) = \sum_{\{U\}} \int_U \sqrt{\det(g_{ij})}\; du_1 du_2,$$

by choosing a partition $\{U\}$ of Σ. Area is independent of choice of partition.

Def. Two metrics $\mathcal{G} = \sum_{i,j} g_{ij} du_i du_j$ and $\mathcal{H} = \sum_{i,j} h_{ij} du_i du_j$ are *conformally equivalent* or simply *conformal* if there exists a function $f(u_1,u_2) > 0$ such that $\mathcal{G} = f^2\mathcal{H}$, i.e., $g_{ij} = f^2 h_{ij}$ for all i,j. f is called the *conformal factor*. The length of every vector at (u_1,u_2) is multiplied by $f(u_1,u_2)$.

An equivalence class of metrics on Σ conformal to one another is called a *conformal structure* on Σ.

Thm. (*Riemann uniformisation & mapping*) Every metric on a connected surface is conformally equivalent to a metric of constant Gaussian curvature $K = \kappa_1\kappa_2$.

Riemann surface theory: every Riemann surface is covered by one of the sphere, the plane, or the Poincaré upper half-plane (hyperbolic geometry).

Classically: a simply connected domain of $\mathbb{C}$ is mapped onto the open disk D by a bijective and holomorphic — hence conformal! — map.

Def. A curve $\beta = x \circ \alpha$ (x is the param. of a surface; α is the arclength param. of a plane curve) is a *geodesic* on x if one of the following equivalent conditions are satisfied:

(i) For $k = 1,2$, $\alpha_k'' + \Gamma_{ij}^k \alpha_i' \alpha_j' = 0$.

(ii) $\beta'' = L_{ij}\alpha_i'\alpha_j'\nu$.

(Here, ν is the unit normal vector, $L_{ij} := \langle \nu, x_{ij} \rangle$ is the second fundamental form II, and Γ_{ij}^k is determined by $x_{ij} = \frac{\partial^2 x}{\partial u_i \partial u_j} = \Gamma_{ij}^1 x_1 + \Gamma_{ij}^2 x_2 + L_{ij}\nu$.)

A *closed geodesic* is defined in the obvious way.

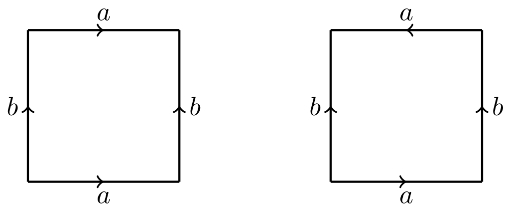
The length of a path $\beta : [a,b] \to \Sigma$ is

$$\mathrm{len}(\beta) := \int_a^b \|\beta'(t)\| \, dt,$$

where $\|v\| = \sqrt{\sum g_{ij} v_i v_j}$.

Def. A (Riemannian) metric is *flat* if its Gaussian curvature K vanishes everywhere.

A closed surface of constant $K = 0$ is topologically either a torus T^2 or a Klein bottle. A flat torus may be identified with $\mathbb{R}^2/\Lambda$, for Λ a 2-dimensional lattice.



2 Topological preliminaries

Def. A *loop* in X is a continuous map $[a,b] \xrightarrow{\gamma} X$ such that $\gamma(a) = \gamma(b)$.

A loop is *contractible* if it can be continuously 'shrunk down' to a point. Equivalently, $S^1 \to X$ extends to a map $D^2 \to X$: the loop does not bound a 'hole'. (Technical details in Chapters 0-1 of Hatcher, *Algebraic Topology*.) X is *simply connected* if every loop in X is contractible. Example: $\mathbb{R}^n - \{0\}$ for $n > 2$.

Def. The set of homotopy equivalence classes of loops in X forms a group under concatention of loop (representatives), called the *fundamental group* $\pi_1(X)$. If spaces X and Y are homotopy equivalent, then $\pi_1(X) \cong \pi_1(Y)$.

Examples:

1) $\pi_1(S^1) \cong \mathbb{Z}$, but $\pi_1(S^n) \cong 0$ for $n > 1$.
2) $\pi_1(S^1 \vee S^1) \cong F_2$, the free group on two generators.
3) $\pi_1(T^2) = \pi_1(S^1 \times S^1) = \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2$.

$(*)$ **Thm.** Every homotopy class of loops in a closed manifold contains a closed geodesic.

Thm. (*Gauss–Bonnet*) Every closed (orientable) surface satisfies

$$\int_\Sigma K(u_1, u_2) \sqrt{\det(g_{ij})} \, du_1 du_2 = 2\pi\chi(\Sigma),$$

where $\chi = 2 - 2g$ is the *Euler characteristic* of Σ and g is the genus of Σ, thought of as the number of 'holes'. Examples: $g = 0$ for S^2; $g = 1$ for T^2.

3 Loewner's torus inequality

Def. Let $(M, \mathcal{G})$ be a Riemannian manifold. The *systole* of $\mathcal{G}$ is $\mathrm{sys}\pi_1(\mathcal{G}) := \inf_\beta \mathrm{len}(\beta)$, defined as the infimum of lengths of β taken over noncontractible loops in $(M, \mathcal{G})$ (`draw a torus and a genus 2 surface with systoles highlighted`).

By $(*)$, the infimum is always attainable; β_{sys} is necessarily a closed geodesic.

Thm. (Loewner, 1949) Every Riemannian metric $\mathcal{G}$ on the torus T^2 satisfies

$$\mathrm{sys}\pi_1(\mathcal{G})^2 \le \frac{2}{\sqrt{3}} \mathrm{area}(\mathcal{G}).$$

Equality is obtained precisely for the flat torus $\mathbb{C}^2/\Lambda$, where the lattice $\Lambda = \mathbb{Z}[\omega]$ is the Eisenstein integers generated by the integral basis $1, \omega = e^{2\pi i/3}$ (`draw a graph of` $\Lambda \subset \mathbb{C}$`, highlighting the fundamental parallelogram`).

The constant $\gamma_2 = 2/\sqrt{3}$ is the *Hermite constant* for $d = 2$, related to the hexagonal packing being the densest circle packing in $\mathbb{R}^2$.

Proof: The proof relies on the conformal representation $\phi : T_{\text{flat}} \to (T^2, \mathcal{G})$, where T_{flat} is a flat torus. Such a representation is possible by the uniformisation theorem. Let f be the conformal factor so that

$$\mathcal{G} = f^2(du_1^2 + du_2^2),$$

where $du_1^2 + du_2^2$ is (locally) the flat metric for T_{flat} obtained earlier.

Let ℓ_0 be any closed geodesic in T_{flat}, and ℓ_s the family of geodesics parallel to ℓ_0. Parametrise ℓ_s by a circle of length σ, such that

$$\sigma \cdot \ell_0 = \text{area}(T_{\text{flat}}).$$

(For ℓ, we slightly abuse notation and mean both the geodesic and its length.)

$T_{\text{flat}} \to S^1$ is a Riemannian submersion. We have

$$\text{area}(T^2) = \int_{T_{\text{flat}}} f^2 \, dA = \int_{S^1} \left(\int_{\ell_s} f^2 \, dt \right) ds,$$

where the last equality 'separating' the double integral follows from Fubini's theorem from analysis.

By the Cauchy–Schwarz inequality,

$$\left(\int_{\ell_s} f \, dt \right)^2 \leq \int_{\ell_s} 1 \, dt \int_{\ell_s} f^2 \, dt = \ell_0 \int_{\ell_s} f^2 \, dt.$$

Therefore, we have that

$$\text{area}(T^2) \geq \frac{1}{\ell_0} \int_{S^1} (\text{len}\, \phi(\ell_s))^2 \, ds,$$

and so there must be an $s_0 \in S^1$ such that $\text{area}(T^2) \geq \frac{\sigma}{\ell_0}(\text{len}\, \phi(\ell_{s_0}))^2$, so that $\text{len}\, \phi(\ell_{s_0}) \leq \ell_0$. We have thus reduced the proof to the flat case, which can easily be examined visually (`draw the fundamental parallelogram of` $T_{\text{flat}} = \mathbb{C}/\mathbb{Z}[\omega]$ `with area` $\sqrt{3}/2$ `and the circle` $\arg z = \pi/3$ `on its surface, showing visually that this achieves the minimum` $\sigma/\ell_0 = 1$ `attainable`). □

4 Other systolic results

There is also *Pu's inequality* for a Riemann surface M homeomorphic to the real projective plane $\mathbb{R}P^2$:

Thm. (Pu, 1950)

$$\text{sys}\pi_1(M)^2 \leq \frac{\pi}{2} \text{area}(M).$$

Again, this bound is sharp.

The *systolic ratio* (SR) for $\mathbb{R}P^2$ is thus $\pi/2$. SR is $2/\sqrt{3}$ for the torus T^2 as proven above; $\pi/2^{3/2}$ for the Klein bottle $\mathbb{R}P^2 \# \mathbb{R}P^2$ (Bavard, 1986); and $(1 + \sqrt{2})/3 = 0.8047... < \text{SR} \leq 2/\sqrt{3}$ for the surface of genus 2 (Katz–Sabourau, 2006). Conjecturally, the SR for surfaces of positive genus g is $2/\sqrt{3}$; this is known to be true for $g \geq 20$ (Katz–Sabourau, 2005).

For higher dimensions, we have *Gromov's systolic inequality* for the class of 'essential manifolds':

Thm. (Gromov, 1983)

$$\text{sys}\pi_1(M)^n \leq C_n \,\text{vol}\, M$$

C_n depends only on the dimension of the manifold M.

For the notion of a *2-systole*, the infimum of areas of 2-cycles representing $\mathbb{C}P^1 \subset \mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$:

Thm. (Gromov, 1981)

$$\text{stsys}_2^n \geq n! \,\text{vol}(\mathbb{C}P^n).$$

Applications: There is a link to quantum error correcting codes which goes by the name of $\mathbb{Z}_2$-systolic freedom, due to Freedman and Hastings (circa 2000). Fetaya (2011) provides a link to homological error correcting codes. Unfortunately, whether qeccs and heccs are the same things or what said links even are, I've absolutely no clue.