MH4921 Algebraic Topology

Sheaf cohomology, Serre duality, and the Riemann–Roch theorem

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Introduction: a bird's eye view

High-level view:

A sheaf is an object that tracks data (here, abelian groups) on open sets of a space and provides a way to glue these data together globally. Exactness of sheaves induces long exact sequences on sheaf cohomology. We show that de Rham cohomology, defined by differential forms, agrees with singular cohomology over complex coefficients.

We then define divisors as formal sums of codimension 1 subvarieties (so for Riemann surfaces, just points), and prove the Serre duality theorem for sheaves of divisors inductively.

Finally, we culminate by proving the Riemann–Roch theorem, relating the complex analysis of meromorphic functions as vector spaces on a Riemann surface and the topology (genus) of said Riemann surface. A striking application is presented: part of the proof of the elliptic curve group law.

1. Preliminaries on sheaves
2. Serre duality
3. The Riemann–Roch theorem

Sheaves

Def. A *presheaf of abelian groups* $\mathcal{F}$ on a topological space $X, \mathcal{T}$ is a collection of abelian groups $(\mathcal{F}(U))_{U \in \mathcal{T}}$ along with group homomorphisms $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ whenever U, V are open and $V \subset U$, such that $\rho^U_W = \rho^V_W \circ \rho^U_V$ and $\rho^U_U = \mathrm{id}_U$.

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Here, we stipulate that each group $\mathcal{F}(U)$ is a group of functions on U, so ρ^U_V is the restriction map $f|_V = \rho^U_V(f)$.

Def. A *sheaf of abelian groups* is a presheaf such that

i) Given a collection of open sets $(U_i)_{i \in I}$ with $U = \bigcup_{i \in I} U_i$ and an element f_i in each $\mathcal{F}(U_i)$, if $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

ii) If $f, g \in \mathcal{F}(U)$ and $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$.

Sheaves

Examples. On a Riemann surface:

- $\mathcal{O}$: (sheaf of) holomorphic functions
- Ω: holomorphic 1-forms
- $\underline{\mathbb{C}}$: locally constant functions with values in $\mathbb{C}$
- $\mathcal{E}$: differentiable functions — *structure sheaf of a differentiable manifold*
- $\mathcal{E}^{(1)}$: differentiable 1-forms
- $\mathcal{E}^{1,0}$: differentiable 1-forms locally resembling $f\,dz$, i.e., having no $d\bar{z}$ term
- $\mathcal{Z}$: closed 1-forms

Sheaves

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Suppose $f \in \mathcal{F}(U_1), g \in \mathcal{F}(U_2)$, and $[f, x], [g, x]$ are their corresponding elements in the stalk $\mathcal{F}_x$. If $[f, x] = [g, x]$, then there exists a neighbourhood $V \subset U_1 \cap U_2$ of x such that $f|_V = g|_V$. (This follows from properties of direct limits.)

Cohomology of sheaves

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Def. Given an open cover $\mathcal{U} = (U_i)_{i \in I}$ and a sheaf $\mathcal{F}$, we define the *zeroth cochain group* $C^0(\mathcal{U}, \mathcal{F}) := \prod_i \mathcal{F}(U_i)$. C^0 is a collection of local sections of the sheaf $\mathcal{F}$. An element of which is denoted (f_i).

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Def. For any $n \in \mathbb{N}$, define the n^{th} *cochain group* as

$$C^n(\mathcal{U},\mathcal{F}) := \prod_{(i_0,\cdots,i_n)\in I^{n+1}} \mathcal{F}(U_{i_0}\cap\cdots\cap U_{i_n}).$$

An n-cochain is then denoted $(f_{i_0,\dots,i_n})$, indexed by a multi-index.

Cohomology of sheaves

It is natural to define the *coboundary operator* δ: an element of C^0 maps to an element of C^1 by taking the difference of the restrictions of f_i and f_j to $U_i \cap U_j$.

Def. $\delta : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ is defined on each 'coordinate' by $\partial(f_i) = (g_{ij})$, where $g_{ij} = f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j}$. Similarly, $\delta : C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$ is defined by $\delta(f_{ij}) = (g_{ijk})$, $g_{ijk} = f_{ij}|_{U_{ijk}} + f_{jk}|_{U_{ijk}} + f_{ki}|_{U_{ijk}}$, each restricted to $U_{ijk} := U_i \cap U_j \cap U_k$.

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Intuitively, δ measures the failure of a 0-cochain to paste together into a global section of $\mathcal{F}$. If $\delta(f_i) = 0$, then $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, so by the first sheaf axiom the (f_i) can be 'pasted' together into a function $f \in \mathcal{F}(X)$. Such elements of C^0 are called *closed cochains*, and the group of closed cochains is denoted $Z^0(\mathcal{U}, \mathcal{F}) = \ker \delta_0$. Similarly, $Z^1 = \ker \delta_1$.

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Define also the *coboundary image set* $B^1(\mathcal{U}, \mathcal{F}) = \operatorname{im} \delta_0$. $B^0 := 0$.

Cohomology of sheaves

One can check that $\delta^2 = 0$. This makes the cochain groups a cochain complex:

$$C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \cdots$$

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Def. The n^{th} *cohomology group* is $H^n(\mathcal{U}, \mathcal{F}) := Z^n(\mathcal{U}, \mathcal{F})/B^n(\mathcal{U}, \mathcal{F}) = \ker \delta_i / \operatorname{im} \delta_{i-1}$.

Since $B^0 = 0$, we have $H^0 = Z^0$, the global sections of the sheaf $\mathcal{F}$.

Cohomology of sheaves

For two open covers $\mathcal{B} = (B_j)_{j \in J}$ and $\mathcal{U} = (U_i)_{i \in I}$, $\mathcal{B}$ is *finer* than $\mathcal{U}$, written $\mathcal{B} < U$, if every B_j is contained in some $U_i = U_{\tau(j)}$ ($\tau : J \to I$ is just a mapping of indices).

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This allows us to define maps $\tau^{\mathcal{U}}_{\mathcal{B}} : Z^1(\mathcal{U}, \mathcal{F}) \to Z^1(\mathcal{B}, \mathcal{F})$ and $B^1(\mathcal{U}, \mathcal{F}) \to B^1(\mathcal{B}, \mathcal{F})$, which together induce $\tau^{\mathcal{U}}_{\mathcal{B}} : H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{B}, \mathcal{F})$ on cohomology. We omit proofs of the following facts: $\tau^{\mathcal{U}}_{\mathcal{B}}$ does not depend on choice of τ above; and it is injective.

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From these facts, we may define $H^1(X, \mathcal{F}) := \varinjlim H^1(\mathcal{U}, \mathcal{F})$, the direct limit being taken over all open covers $\mathcal{U}$ of X. We will usually just write $H^1(\mathcal{F})$.

$$\boxed{0 \to H^0(\mathcal{F}) \xrightarrow{\delta} H^1(\mathcal{F}) \xrightarrow{\delta} H^2(\mathcal{F}) \to \cdots}$$

Exact sequences of sheaves

Def. A *sheaf homomorphism* $\alpha : \mathcal{F} \to \mathcal{G}$ is a collection of homs $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that the maps commute with restriction maps ρ — for all open sets $V \subset U$:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\alpha_U} & \mathcal{G}(U) \\ \rho^U_V \downarrow & & \downarrow \rho^U_V \\ \mathcal{F}(V) & \xrightarrow[\alpha_V]{} & \mathcal{G}(V) \end{array}$$

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α induces homs of stalks $\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x$. A hom of sheaves is mono if it is mono (injective) on every stalk; and epi if epi (surjective) on every stalk. A sequence $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ is exact if, for every x, $\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$ is exact.

Exact sequences of sheaves

Lemma 1.1

If $\alpha : \mathcal{F} \to \mathcal{G}$ is a sheaf monomorphism, then every $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is also a monomorphism.

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However, the same statement for sheaf epimorphisms fails to be true. Example: $\mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times$, where $\exp(f) = e^{2\pi i f}$. Locally (at stalks, i.e., neighbourhoods of points in $\mathbb{C}^\times$), we see that exp is indeed invertible. However, on general open sets $U \subset \mathbb{C}^\times$ (including $\mathbb{C}^\times$ itself), exp is not invertible — the logarithm is multivalued, so in particular $\mathrm{id} \in \mathcal{O}^\times(\mathbb{C}^\times)$ has no preimage under exp.

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Not all hope is lost:

Exact sequences of sheaves

Lemma 1.2

If $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ *is an exact sequence of sheaves, then for any open set* U*,* $0 \to \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$ *is also exact.*

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This sets up for a remarkable fact:

Exact sequences of sheaves

Lemma 1.3

If $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ *is a short exact sequence of sheaves, then*

$$0 \to H^0(\mathcal{F}) \xrightarrow{\alpha} H^0(\mathcal{G}) \xrightarrow{\beta} H^0(\mathcal{H}) \xrightarrow{\delta^*} H^1(\mathcal{F}) \xrightarrow{\alpha} H^1(\mathcal{G}) \xrightarrow{\beta} H^1(\mathcal{H}) \to \cdots$$

is a long exact sequence. δ^* *is called the* connecting homomorphism.

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From this, we can take relative cohomology, and compute $H^1(\mathcal{F}) = H^0(\mathcal{H})/\beta(H^0(\mathcal{G}))$.

De Rham cohomology

Def. For functions, $df = f_z dz + f_{\bar{z}} d\bar{z}$. Define $\partial f := f_z dz$ and $\overline{\partial} f := f_{\bar{z}} d\bar{z}$ for functions, extending to differential forms in the obvious way. Clearly $d = \partial + \overline{\partial}$.

$\mathcal{E}^{p,q}$ is mapped to $\mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1}$ by d; to $\mathcal{E}^{p+1,q}$ by ∂; and to $\mathcal{E}^{p,q+1}$ by $\overline{\partial}$. A function is holomorphic iff $\overline{\partial} f = 0$ (Cauchy–Riemann).

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Def. The n^{th} *de Rham cohomology group* is $H^n_{\mathrm{dR}}(X) := H^{n-1}(\mathcal{Z})/d(H^{n-1}(\mathcal{E}))$.

(Recall that $\mathcal{Z}$ and $\mathcal{E}$ are the sheaves of closed 1-forms and differentiable functions respectively; also that $\underline{\mathbb{C}}$ is the sheaf of locally constant complex-valued functions.)

De Rham cohomology

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Proof: This follows from the exactness of the sequence

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There is also the Dolbeault theorem for Dolbeault cohomology in bidimension (p, q) involving $\overline{\partial}$.

1 Preliminaries on sheaves

2 Serre duality

3 The Riemann–Roch theorem

Divisors

In full generality, Serre duality for coherent sheaves is a statement involving the dualising sheaf ω_X on a (Cohen–Macaulay) scheme X:

$$\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \cong H^{n-i}(X, \mathcal{F})^*.$$

ω_X corresponds to the canonical divisor K in the specific case of Riemann surfaces. We will not concern ourselves with this vast generalisation and instead restrict ourselves to the case of *Riemann surfaces*, i.e., 1-dimensional complex manifolds.

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Def. A *divisor* on a compact Riemann surface X is a finite formal sum $\sum_i c_i p_i$, where $c_i \in \mathbb{Z}$ and $p_i \in X$. Denote the set of divisors on X by $\mathrm{Div}(X)$.

Divisors

The *divisor of a (meromorphic) function* $f \in \mathcal{M}(X)$ is the divisor $(f) := \sum_p \operatorname{ord}_p(f)p$. In the same way that ideals generated by a single element are called principal, a divisor which is the divisor of a function is called *principal*; as a set, $\operatorname{Prin}(X)$.

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$\operatorname{Cl}(X) := \operatorname{Div}(X)/\operatorname{Prin}(X)$ is called the *divisor class group* of X. A divisor is *effective* if $c_i \geq 0$ for all i. We can add/subtract divisors by adding/subtracting their formal sums. Finally, we write $D \geq D'$ if $D - D'$ is effective.

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The *degree* of a divisor is the map $\deg : \operatorname{Div}(X) \to \mathbb{Z}; \sum c_i p_i \mapsto \sum c_i$. Using a fact from complex analysis: If D is principal, then $\deg D = 0$, since on a compact Riemann surface, the orders of zeroes and poles sum to 0.

Divisors

Given two meromorphic 1-forms ω_1, ω_2, choosing some coordinate chart U_z, we note that their ratio $\omega_1/\omega_2 = f_1 dz/f_2 dz = f_1/f_2$. This latter quantity is invariant under change of coordinate charts, so any two meromorphic 1-forms lie in the same divisor class, which we shall represent with the *canonical divisor* $K_X = K$.

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Def. For a divisor D, define the sheaf $\mathcal{O}_D$ by $\mathcal{O}_D(U) := \{f \in \mathcal{M}(U) : (f) \geq -D\}$.

Divisors

Some basic facts:

If $D = 0$, $\mathcal{O}_D = \mathcal{O}$. If $D \leq D'$, then $\mathcal{O}_D(U) \subset \mathcal{O}_{D'}(U)$ for all open sets U.

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If D and D' lie in the same divisor class, then $\mathcal{O}_D \cong \mathcal{O}_{D'}$.

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Lemma 2.2

$\mathcal{O}_{D+K} \cong \Omega_D$.

Divisors

Def. The *skyscraper sheaf* $\mathbb{C}_p$ is defined by $\mathbb{C}_p(U) := \begin{cases} \mathbb{C} & \text{if } p \in U \\ \{0\} & \text{if } p \notin U \end{cases}$.

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Lemma 2.3

i) $H^0(X, \mathbb{C}_p) \cong \mathbb{C};$

ii) $H^1(X, \mathbb{C}_p) \cong 0.$

Divisors

Proof: (i) Any open cover $\mathcal{U}$ of X can be refined so that only one U_i contains p. Then $H^0(\mathcal{U}, \mathbb{C}_p) \cong Z^0(\mathcal{U}, \mathbb{C}_p) \cong \mathbb{C}$, and pass to the direct limit.

Divisors

Proof: (i) Any open cover $\mathcal{U}$ of X can be refined so that only one U_i contains p. Then $H^0(\mathcal{U}, \mathbb{C}_p) \cong Z^0(\mathcal{U}, \mathbb{C}_p) \cong \mathbb{C}$, and pass to the direct limit.

(ii) For any open cover $\mathcal{U}$, we can again refine to an open cover where p is contained in only one of the open sets. No intersection $U_i \cap U_j \ni p$, so $Z^1(\mathcal{U}, \mathbb{C}_p) = \{0\}$. Refinement produces an injection, so $Z^1 = \{0\}$ for all open covers; thus, $H^1(\mathcal{U}, \mathbb{C}_p) \cong 0$, and once again take the direct limit. □

Divisors

Finally, define $\beta : \mathcal{O}_{D+p} \to \mathbb{C}_p$ like so. If $f \in \mathcal{O}_{D+p}(U)$ and $p \in U$, pick a coordinate chart (V, z) with $V \subset U$, $z(V) = D^2$, and $z(p) = 0$. Locally about p, $f(z) = \sum_{n=-k-1}^{\infty} a_n z^n$, where D has value k and $D + p$ has value $k + 1$ at p. Set $\beta(f) = a_{-k-1}$. Otherwise, if $p \notin U$, set $\beta(f) = 0$.

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$$0 \to \mathcal{O}_D \xrightarrow{i} \mathcal{O}_{D+p} \xrightarrow{\beta} \mathbb{C}_p \to 0$$

is thus a short exact sequence of sheaves, inducing a long exact sequence on cohomology:

$$\boxed{0 \to H^0(\mathcal{O}_D) \xrightarrow{i} H^0(\mathcal{O}_{D+p}) \xrightarrow{\beta} \mathbb{C} \xrightarrow{\delta^*} H^1(\mathcal{O}_D) \xrightarrow{i} H^1(\mathcal{O}_{D+p}) \xrightarrow{\beta} 0.}$$

Serre duality for Riemann surfaces

$$0 \to H^0(\mathcal{O}_D) \to H^0(\mathcal{O}_{D+p}) \to \mathbb{C} \to H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{D+p}) \to 0$$

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By Lemma 2.2, we can rewrite this as

$$0 \to H^0(\Omega_{D-K}) \to H^0(\Omega_{D-K+p}) \to \mathbb{C} \to H^1(\Omega_{D-K}) \to H^1(\Omega_{D-K+p}) \to 0 \qquad (1)$$

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Replacing D with $K - D - p$, we have

$$0 \to H^0(\mathcal{O}_{K-D-p}) \to H^0(\mathcal{O}_{K-D}) \to \mathbb{C} \to H^1(\mathcal{O}_{K-D-p}) \to H^1(\mathcal{O}_{K-D}) \to 0,$$

and so the long exact sequence on the cohomology groups viewed as dual vector spaces:

$$0 \to H^1(\mathcal{O}_{K-D})^* \to H^1(\mathcal{O}_{K-D-p})^* \to \mathbb{C} \to H^0(\mathcal{O}_{K-D})^* \to H^0(\mathcal{O}_{K-D-p})^* \to 0 \qquad (2)$$

Serre duality for Riemann surfaces

Finally, we can combine (1) and (2) into the diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & H^0(\Omega_{D-K}) & \longrightarrow & H^0(\Omega_{D-K+p}) & \longrightarrow & \mathbb{C} & \longrightarrow & H^1(\Omega_{D-K}) & \longrightarrow & H^1(\Omega_{D-K+p}) & \longrightarrow & 0 \\
\downarrow & & \downarrow i & & \downarrow i & & \downarrow & & \downarrow i & & \downarrow i & & \downarrow \\
0 & \to & H^1(\mathcal{O}_{K-D})^* & \to & H^1(\mathcal{O}_{K-D-p})^* & \to & \mathbb{C} & \to & H^0(\mathcal{O}_{K-D})^* & \to & H^0(\mathcal{O}_{K-D-p})^* & \to & 0
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where the i's are monomorphisms (proof omitted) arising from the functional Res from complex integration.

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where the i's are monomorphisms (proof omitted) arising from the functional Res from complex integration.

Inductively, assume that $H^0(\Omega_{-D'}) \cong H^1(\mathcal{O}_{D'})^*$, where $D' := K - D$. Then the leftmost i is an isomorphism. By the five lemma, the left inner i is an isomorphism. Similarly, the rightmost i is an isomorphism as well.

Serre duality for Riemann surfaces

We have shown that $H^0(\Omega_{-D'}) \cong H^1(\mathcal{O}_{D'})^*$ implies $H^0(\Omega_{-(D'+p)}) \cong H^1(\mathcal{O}_{D'+p})^*$; a symmetric argument for $-(D'-p)$ and $D'-p$ concludes the inductive step.

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Finally, Serre duality holds in the base case $D' = K - D = 0$ — i.e., $H^0(\Omega) = H^1(\mathcal{O})^*$ — by an argument involving Stokes' theorem and an inner product of 1-forms (proof omitted). Thus,

Serre duality for Riemann surfaces

Theorem 2.4 (Serre duality)

$$H^0(\Omega_{-D}) \cong H^1(\mathcal{O}_D)^*.$$

1. Preliminaries on sheaves
2. Serre duality
3. The Riemann–Roch theorem

Genus

Def. The *genus* of a Riemann surface X is $g = \frac{1}{2} \dim H^1(X)$.

Since $H^1(X) \cong H^1_{\mathrm{dR}}(X) \cong H^0(\Omega) \oplus \overline{H^0(\Omega)}$ and $\overline{H^0(\Omega)} \cong H^1(\mathcal{O})$, we have that

$$g = \frac{1}{2} \dim H^1(X) = \dim H^0(\Omega) = \dim H^1(\mathcal{O}),$$

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The dimension of the space of holomorphic 1-forms $H^0(\Omega)$ is also known as the *geometric genus* of X, so both definitions of genus agree.

Riemann–Roch theorem

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Theorem 3.1 (Riemann–Roch, first version)

Let D be a divisor on a compact Riemann surface of genus g. Then

$$\dim H^0(\mathcal{O}_D) - \dim H^1(\mathcal{O}_D) = 1 - g + \deg(D).$$

Riemann–Roch theorem

Proof: Suppose $D = 0$. Then $\mathcal{O}_D = \mathcal{O}$, $H^0(\mathcal{O})$ consists of constant functions so has dimension 1, $H^1(\mathcal{O}) = g$ by definition, and $\deg(D) = 0$. Thus, the LHS and RHS agree, being $1 - g$.

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Now suppose the identity holds for a divisor D. We have again the long exact sequence

$$0 \to H^0(\mathcal{O}_D) \to H^0(\mathcal{O}_{D+p}) \xrightarrow{\beta} \mathbb{C} \xrightarrow{\delta^*} H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{D+p}) \to 0.$$

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Separate this into two sequences, checking that they are short exact:

$$0 \to H^0(\mathcal{O}_D) \to H^0(\mathcal{O}_{D+p}) \xrightarrow{\beta} \operatorname{im}\beta \to 0$$

$$0 \to \mathbb{C}/\operatorname{im}\beta \xrightarrow{\delta^*} H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{D+p}) \to 0$$

Riemann–Roch theorem

We have $\dim H^0(\mathcal{O}_{D+p}) = \dim H^0(\mathcal{O}_D) + \dim \operatorname{im} \beta$ and $\dim H^1(\mathcal{O}_{D+p}) = \dim \mathbb{C}/\operatorname{im} \beta + \dim H^1(\mathcal{O}_{D+p})$ from the first and second short exact sequences, respectively.

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Adding both equations up, we get

$$\begin{aligned} \dim H^0(\mathcal{O}_{D+p}) + \dim H^1(\mathcal{O}_{D+p}) &= \dim H^0(\mathcal{O}_D) + \dim H^1(\mathcal{O}_D) + 1 \\ &= (1 - g + \deg(D)) + 1 \\ &= 1 - g + \deg(D+p). \end{aligned}$$

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Argue analogously for $D - p$. This completes our induction. $\square$

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Let D be a divisor on a compact Riemann surface of genus g. Then

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This, of course, follows from $\Omega_{-D} = \mathcal{O}_{K-D}$ and thus $\dim H^1(\mathcal{O}_D) = \dim H^1(\mathcal{O}_D)^* = \dim H^0(\Omega_{-D}) = \dim H^0(\mathcal{O}_{K-D})$.

Applications

For brevity, denote the dimension of the vector space of meromorphic functions f on X with $(f) \geq -D$ by $\ell(D)$. The second form of Riemann–Roch is thus simply

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As an immediate consequence, we can calculate the degree of the canonical divisor K:

$$\begin{aligned}\deg(K) &= \dim H^0(\mathcal{O}_K) - \dim H^0(\mathcal{O}_{K-K}) + g - 1 \\ &= \dim H^0(\Omega) - \dim H^0(\mathcal{O}) + g - 1 \\ &= g - 1 + g - 1 \\ &= 2g - 2 = -\chi(X),\end{aligned}$$

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Applications

The theorem also leads to the classification of Riemann surfaces by their possible universal covers: the sphere S^2, the complex plane $\mathbb{C}$, or the Poincaré half-plane model $\mathcal{H}$ (equivalently, the disk as a model of the hyperbolic plane).

Applications

Def. An *elliptic curve* is the solution set $E := \{[x : y : z] \in \mathbb{C}P^2 : E(x, y, z) = 0\}$, where $E(x, y, z) = y^2z - x^3 - axz^2 - bz^3$, $a, b \in \mathbb{C}$, and $4a^3 + 27b^2 \neq 0$.

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The proof of the group law depends on Riemann–Roch.

Applications

The goal is to show that the always-injective *Abel–Jacobi map* $J : E \to \mathrm{Cl}(E); P \mapsto (P) - (O)$ is also surjective, and hence E inherits the group structure from $\mathrm{Cl}(E) = \mathrm{Pic}^0(E)$ (in this context, the divisor class group is exactly a more general construction, the *Picard group*).

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Let $D \in \mathrm{Div}(E)$. By Riemann–Roch, $\ell(D + (O)) = 1$, so if $f \in H^0(\mathcal{O}_{D+(O)})$ is not constant, we have $(f) = -D - (O) + (P)$ for some P. Since $\deg(D) = \deg((f)) = 0$, $\deg((P)) = \deg((O)) = 1$, so $P \in E$ and is sent to $(P) - (O)$, in the same divisor class as D.

Thank you!

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